# Analytic Stacks

Shuai Zong

Summer 2024

**Status.** These notes develop light condensed sets and basic analytic rings. Advanced solidification, liquid vector spaces and analytic Riemann–Hilbert drafts remain in supporting; their incomplete arguments are not results proved here.

# Contents


**1 Light Condensed Sets** **1**

1.1 Condensed Set . . . . . 1

1.2 Condensed abelian group . . . . . 4

**2 Analytic Ring** **7**

2.1 Huber pair . . . . . 7

2.1.1 Huber rings . . . . . 7

2.1.2 Continuous Valuations . . . . . 8

2.2 Analytic Rings . . . . . 9

# Chapter 1

# Light Condensed Sets

## 1.1 Condensed Set

**Definition 1.1.1.** *The site* ProFin *is defined as the category of profinite sets with finite jointly surjective families of maps as covers. A condensed set is a sheaf of sets on the site* ProFin*, the category of condensed set is denoted by* Cond*. And similarly, we can define the category* CondAb *of the condensed abelian group.*

In particular, the topology on ProFin is generated by finite disjoint union and all surjections, so a condensed set is a functor $T$: ProFin$^{op}$ $\to$ Set satisfying $T(\emptyset) = *$ and the following two (sheaf) conditions:

1. For any profinite sets $S_1, S_2$, the natural map

$$T(S_1 \sqcup S_2) \to T(S_1) \times T(S_2)$$

   is a bijection.

2. For any surjection $S' \to S$, the following sequence is exact

$$T(S) \to T(S') \rightrightarrows T(S' \times_S S').$$

The category Cond is not well defined, since there is some set theoretical issues that the category ProFin is not small. The problem can be fixed by define the category Cond as the colimit of catgeory of condensed sheaf on some space with bounded cardinality, but in this notes we will work on the light profinite sets.

**Proposition 1.1.2.** *Let $S$ be a profinite set, the following are equivalent:*

1. *$S$ is metrizable,*
2. *$S$ is second-countable,*
3. *$S$ can be wirtten as a sequential limit of finite sets,*
4. *$\mathrm{Cont}(S, \mathbb{Z})$ is countable.*

*Proof.* $(1) \Leftrightarrow (2)$ : By Urysohn's metrization theorem, a compact Hausdorff space is metrizable if and only if it is second-countable.

$(2) \Leftrightarrow (3)$ : If $S = \lim_{\mathbb{N}} S_i$ for some finite set $S_i$, then the fiber of the projection $S \to S_n$ defines a countable basis of the topology. If $S$ is second countable, the finite quotients of $S$ gives a countable projective system.

$(3) \Rightarrow (4)$ : If we have $S = \lim_{\mathbb{N}} S_i$ for some finite set $S_i$, then $\mathrm{Cont}(S, \mathbb{Z}) = \mathrm{colim}\, \mathrm{Cont}(S_i, \mathbb{Z})$, which is countable.

$(4) \Rightarrow (2)$ : If $\mathrm{Cont}(S, \mathbb{Z})$ is countable, then $\mathrm{Cont}(S, \mathbb{F}_2)$ is countable. Since clopen subspaces of $S$ are in bijection with the elements of $\mathrm{Cont}(S, \mathbb{F}_2)$, we get that $S$ is second-countable. □

**Definition 1.1.3.** *A profinite set is called light if it satisfies the above equivalent conditions, we will abuse notation and refer to the category of light profinite set as* ProFin*.*

There are some nice features that are special to the category of light profinite sets.

**Proposition 1.1.4.** *The category of light profinite sets has all countable limits, and the sequential limits of surjections are surjections.*

*Proof.* By 1.1.2, the countable limits clearly exist. Let $S = \lim_n S_n$ be a sequential limit of surjections, i.e. all maps $S_{n+1} \to S_n$ are surjections, then clearly $S \to S_n$ is surjective for all $n$. □

**Proposition 1.1.5.** *Let $S$ be a light profinite set and let $U \subset S$ be an open subset, then $U$ is a countable disjoint union of light profinite sets.*

*Proof.* We may write $S = \lim S_n$ with the projections $\pi_n \colon S \to S_n$ and set $Z = S \backslash U$, then $Z = \lim Z_n$, where $Z_n = \mathrm{Im}(Z \to S_n)$. We can get

$$U = \bigcup_n \pi_n^{-1}(S_n \backslash Z_n) = \bigsqcup_n \pi_{n+1}^{-1}(S_{n+1} \backslash Z_{n+1}) \backslash \pi_n^{-1}(S_n \backslash Z_n).$$

□

**Proposition 1.1.6.** *Every nonempty light profinite set is an injective object in the category of light profinite sets.*

*Proof.* Consider a lifting problem

$$\begin{array}{ccc} X & \xhookrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & \swarrow{\scriptstyle h} & \\ S & & \end{array}.$$

The map $f$ is a closed immersion, we can write it as a sequential limit of injections between finite sets

$$f = \lim_n f_n, \qquad f_n \colon X_n \to Y_n.$$

We can write the map $g$ as a sequential limit of finite set $\lim_n g_n \colon X_{k_n} \to S_n$. After taking a subsequence we may assume $k_n = n$, then we can inductively construct the map $h_n \colon Y_n \to S_n$ and take the limit $h = \lim h_n$. □

**Example 1.1.7.** *There are two key examples of light profinite set:*

1. *The set $\mathbb{N} \cup \{\infty\}$. For a condensed set $T$, the set $T(\mathbb{N} \cup \{\infty\})$ is heuristically the space of convergence sequences with fixed limit.*

2. *The Cantor set $\prod_{\mathbb{N}} \{0, 1\}$, it is the "universal" light condensed set by following proposition.*

**Proposition 1.1.8.** *A nonempty profinite set is light if and only if it admits a surjection from the Cantor set.*

*Proof.* Let $S = \lim_n S_n$ be a light profinite set, then for each $S_n$, there exists some postive integer $k_n$ such that there exists a surjection $\prod_0^{k_n} \{0, 1\} \to S_n$, after taking suitable limit we obtain the result. The converse direction is trivial. □

**Remark 1.1.9.** *In this note, we will continue to use the previous notation to represent the light condensed set, i.e. the sheaf on light profinit set, and omit the light everywhere in the text.*

**Example 1.1.10.** *We have a faithful functor from the $T_1$ space to the condensed set:*

$$\begin{aligned} &\mathsf{Top}_{T_1} \to \mathsf{Cond} \\ &\quad X \mapsto \underline{X} := [S \mapsto \mathrm{Cont}(S, X)]. \end{aligned}$$

*When restricted to metrizable compact Hausdorff spaces, this functor is fully faithful and it admits a left adjoint. The left adjoint $T \mapsto T(*)_{top}$ can be constructed as follows.*

*Since every condensed set $T$ can be wirtten as the form $T = \mathrm{colim}_{S_i \in \mathsf{ProFin}} \underline{S_i}$, the set $T(*)$ can be equipped the colimit topology. Or equivalently, the topology on $T(*)$ is given by the quotient topology from $\sqcup_{\alpha \in T(S)} S \to T(*)$, where $S$ denotes the Cantor set.*

**Example 1.1.11.** *Let $\mathbb{R}_d$ denote the real numbers with discrete topology, and $\mathbb{R}$ the real numbers with the natural topology. Then the map $\underline{\mathbb{R}_d} \to \underline{\mathbb{R}}$ admits a non-trivial cokernel $Q$ in the category of condensed abelian group. The underlying abelian group of $Q$ is trivial, but for a general profinite set $S$, one has* [1]

$$Q(S) = \mathrm{Cont}(S, \mathbb{R}) / \mathrm{Cont}(S, \mathbb{R}_d) \neq 0.$$

**Definition 1.1.12.** *In a complete and cocomplete category $\mathcal{C}$, we say that one object $X$ is quasi-compact if for any surjection $\sqcup_{i \in I} Y_i \twoheadrightarrow X$ there exists finite subset $J \subset I$ such that $\sqcup_{i \in J} Y_i \twoheadrightarrow X$, one object $X$ is quasi-separated if for quasi-compact objects $Y, Z$ with morphism $Y \to X, Z \to X$, the fibre product $Y \times_X Z$ is quasi-compact.*

**Theorem 1.1.13.** *The quasi-compact, quasi-separated objects in* Cond *are precisely the metrizable compact Hausdorff spaces.*

*Proof.* It is easy to see that an object $C$ is quasi-compact if and only if there exists a surjection from Cantor set by writing $C$ as the colimit of representiable sheaf; then an object $C$ is quasi-separated if and only if for all map from Cantor set, the fibre product of Cantor set is quasi-compact. Then we obtian the desired result. □

## 1.2 Condensed abelian group

**Theorem 1.2.1.** *The forgetful functor* CondAb $\to$ Cond *admits a left adjonit $X \mapsto \mathbb{Z}[X]$, called the free abelian group generated by a condensed set.*

*Proof.* Set $\mathbb{Z}[X] = (S \mapsto \mathbb{Z}[X(S)])^\sharp$. □

[1] Since we have $H^1(S, M) = 0$ for any profinite set $S$ and discrete abelian group $M$, this formula gives the right result.

**Theorem 1.2.2.** *The category* CondAb *is a Grothendieck abelian category with a natural closed symmetric monoidal structue. Moreover we have*

1. *Countable products are exact,*
2. *Sequential limits of surjections are surjections.*
3. $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]$ *is internally projective.*

*Proof.* $(2) \Rightarrow (1)$ : To check the exactness of countable product, we jush need show it preserves surjections, then for surjections $f_n \colon M_n \to N_n$, we have

$$\prod f_n = \lim_m (\prod_{n \leq m} M_n \times \prod_{n > m} N_n \to \prod M_n).$$

$(2)$ : Trivial.

$(3)$ : It suffices to prove that the space of null sequence $M = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}[\infty]$ is internally projective. Consider the following lifting problem:

$$\begin{array}{ccccc} S & & \xrightarrow{\quad g \quad} & & A \\ \downarrow & & & \nearrow & \downarrow \\ \mathbb{N} \cup \{\infty\} & \longrightarrow & M & \xrightarrow[f]{} & B \end{array}$$

By definition there exits some $S \in$ ProFin cover $\mathbb{N} \cup \{\infty\}$ and fill the diagram. After replacing $S$ by its appropriate closed subset, we may assume that $S \times_{\mathbb{N} \cup \{\infty\}} \mathbb{N} \simeq \mathbb{N}$, and we set $i \colon S_\infty = S \times_{\mathbb{N} \cup \{\infty\}} \{\infty\} \hookrightarrow S$. Since $S_\infty$ is injective, there exists a retraction $r \colon S \to S_\infty$. We claim that $\mathbb{N} \cup \{\infty\} = S \sqcup_{S_\infty} \{\infty\}$, then the following diagram gives a morphism $\tilde{f} \colon M \to A$:

$$\begin{array}{ccc} S_\infty & \hookrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle g - g \circ i \circ r} \\ 0 & \longrightarrow & A \end{array}$$

Then it suffices to check $\tilde{f}$ acturally lifts $f$, which is trivial. □

**Definition 1.2.3.** *For* $X \in$ Cond *and* $M \in$ CondAb*, we define the global section of* $M$ *on* $X$ *as*

$$\Gamma(X, M) := \mathrm{Hom}_{\mathsf{Cond}}(X, M) = \mathrm{Hom}_{\mathsf{CondAb}}(\mathbb{Z}[X], M) \in \mathsf{Ab}.$$

*Then we define the cohomology as the derived global section* $R\Gamma(X.M)$.

**Proposition 1.2.4.** *For any light profinite set $S$ and discrete abelian group $M$, we have $R\Gamma(S, M) = \mathrm{Cont}(S, M)$.*

*Proof.* We compute the cohomology via hypercovers, in particular, it suffices to show that for a cover $S' \to S$ with associated Čech nerve, the complex

$$C = 0 \to \mathrm{Cont}(S', M) \to \mathrm{Cont}(S' \times_S S', M) \to \cdots$$

has no non-trivial cohomology in degree $\geq 1$. We can write the surjection $S' \to S$ as a sequential limit of finite sets with surjective maps, then we get $C = \mathrm{colim}\, C_i$, but every surjection of finite sets splits, we get the result. □

**Proposition 1.2.5.** *For any light compact Hausdorff space $X$ and discrete abelian group $M$, we have $R\Gamma(X, M) = R\Gamma_{top}(X, M)$.*

*Proof.* Consider the adjoint pair $(\pi_*, \pi^*)\colon \mathsf{Cond}_{/X} \rightleftarrows \mathrm{Shv}(X)$

$$\pi^*\mathcal{G} = ((s\colon S \to X) \mapsto \Gamma(S, s^*\mathcal{G}))^\sharp,$$
$$\pi_*\mathcal{F} = U \mapsto \mathrm{Hom}_{\mathsf{Cond}_{/X}}(U, \mathcal{F}).$$

This adjunction induces

$$(R\pi_*, \pi^*)\colon \mathcal{D}^+(\mathrm{Shv}(X, \mathsf{Ab})) \leftrightarrows \mathcal{D}^+(\mathsf{CondAb}_{/X}).$$

It suffices to show that $\pi^*$ is fully faithful, by adjunction, we just need show that for $A \in \mathcal{D}^+(\mathrm{Shv}(X, \mathsf{Ab}))$, the unit map $A \to R\pi_*\pi^*A$ is an isomorphsim. We can check this on stalks, fix $x \in X$,since the open neighborhoods are cofinal with the closed neighborhoods, and note that by picking a simplical hypercover $S_\bullet \to X$, the derived section $R\Gamma(X, M)$ can be computed by $\Gamma(S_\bullet, M)$.

Passing to the filtered colimit over all closed neighborhood $V$ of $x$, one gets the complex $R\Gamma(S_\bullet \times_X x)$, which computes that $R\Gamma(x, M) = M$. So we have

$$(R\pi_*\pi^*M)_x = \mathrm{colim}_V\, R\Gamma(V, M) = M.$$

□

# Chapter 2

# Analytic Ring

## 2.1 Huber pair

First of all, we will review the theory of adic spaces as developed by Huber.

### 2.1.1 Huber rings

**Definition 2.1.1.** *A topological ring $A$ is Huber if $A$ admits an open subring $A_0 \subset A$, such that there exists a finitely generated ideal $I \subset A_0$ with a basis of open neighborhoods of $0$ given by $\{I^n, n \geq 0\}$. Any such $A_0$ is called a ring of definition of $A$.*

Note that $A_0$ is not assumed to be $I$-adically complete, but one can always take the completion of $A$ and $A_0$. [1]

**Example 2.1.2.** *There are three key examples to indicate that adic spoaces encompass schemes, formal schems and rigid spaces.*

1. *Scheme : Any discrete ring $A$ is Huber with any $A_0 \subset A$.*
2. *Formal scheme : An adic ring $A$ is Huber if it has a finitely generated ideal of definition.*
3. *Rigid space : Let $A_0$ be any ring, let $g \in A_0$ be a nonzero-divisor, and let $A = A_0[g^{-1}]$, equipped with the topology making $\{g^n A_0\}$ a basis of open neighborhoods of $0$. Then this ring $A$ is Huber, for example, if $A$ is*

[1] By "complete" we mean classical complete.

*a Banach algebra over a nonarchimedean field $K$, we can take $A_0 \subset A$ to be the unit ball, and $g \in K$ any nonzero element with $|g| < 1$.*

**Definition 2.1.3.** *A subset $S$ of a topological ring $A$ is bounded if for all open neighborhoods $U$ of $0$, there exists an open neighborhood $V$ of $0$ such that $VS \subset U$.*

**Lemma 2.1.4.** *A subring $A_0$ of a Huber ring $A$ is a ring of definition if and only if it is open and bounded.*

**Definition 2.1.5.** *A Huber ring $A$ is Tate if it contains a topological nilpotent unit $g \in A$. A pseudo-uniformizer in $A$ is a topological nilpotent unit.*

The Tate rings are exactly the rings that arise by the construction of example given above.

**Definition 2.1.6.** *Let $A$ be a Huber ring. An element $x \in A$ is power-bounded if $\{x^n, n \geq 0\}$ is bounded. We denote the subring of power-bounded elements by $A^\circ$, and we denote the subring of topological nilpotent elements by $A^{\circ\circ}$.*

**Proposition 2.1.7.** *1. Any ring of definition $A_0 \subset A$ is contianed in $A^\circ$.*

*2. The ring $A^\circ$ is the filtered union of the rings of definition $A_0 \subset A$.*

**Definition 2.1.8.** *A Huber ring $A$ is uniform if $A^\circ$ is bounded, or equivalently $A^\circ$ is a ring of definition.*

**Definition 2.1.9.** *1. Let $A$ be a Huber ring. A subring $A^+ \subset A$ is a ring of integral elements if it is open and integrally closed in $A$, and $A^+ \subset A^\circ$.*

*2. A Huber pair is a pair $(A, A^+)$, where $A$ is Huber and $A^+ \subset A$ is a ring of integral elements.*

## 2.1.2 Continuous Valuations

**Definition 2.1.10.** *A continuous valuation on a topological ring $A$ is a map $|-|\colon A \to \Gamma \cup \{0\}$ into a totally order abelian group $\Gamma$ such that*

1. $|ab| = |a||b|$,
2. $|a + b| \leq \max(|a|, |b|)$,

3. $|1| = 1$,
4. $|0| = 0$,
5. *for all* $\gamma \in \Gamma$ *lying in the image of* $|-|$, *the set* $\{a \in A, |a| < \gamma\}$ *is open.*

**Definition 2.1.11.** *The adic spectrum* $\mathrm{Spa}(A, A^+)$ *is the set of equivalence classes of continuous valuations* $|\cdot|$ *on* $A$ *such that* $|A^+| \leq 1$.

*The topology on* $\mathrm{Spa}(A, A^+)$ *is generated by open subsets of the form* $\{x, |f(x)| \leq |g(x)| \neq 0\}$ *with* $f, g \in A$.

**Theorem 2.1.12.** *The topological space* $\mathrm{Spa}(A, A^+)$ *is spectral.*

**Proposition 2.1.13.** *Let* $(\hat{A}, \hat{A}^+)$ *be the completion of a Huber pair* $(A, A^+)$. *Then their adic spectrum are isomorphic to each other.*

**Proposition 2.1.14.** *Let* $(A, A^+)$ *be a complete Huber pair.*

1. *If* $A \neq 0$ *then* $\mathrm{Spa}(A, A^+)$ *is nonempty.*
2. *One has* $A^+ = \{f \in A, \forall x \in X, |f(x)| \leq 1\}$.
3. *An element* $f \in A$ *is invertible if and only if for all* $x \in X, |f(x)| \neq 0$.

## 2.2 Analytic Rings

**Definition 2.2.1.** *An uncomplete analytic ring is a pair* $A = (A^\triangleright, \mathcal{D}(A))$ *consisting on a light condensed ring* $A^\triangleright$ *and a full subcategory* $\mathcal{D}(A) \subset \mathcal{D}(A^\triangleright)$ *satisfying the following properties:*

1. $\mathcal{D}(A)$ *is stable under arbitary limits and colimits, and there exists a left adjoint of the inclusion.*
2. *For all* $C \in \mathcal{D}(\mathsf{CondAb})$ *and* $M \in \mathcal{D}(A)$, *we have* $R\underline{\mathrm{Hom}}(C, M) \in \mathcal{D}(A)$.
3. *The left adjoint sends connective objects to connective objects. In particular,* $\mathcal{D}(A)$ *has a natural t-structure induced from* $\mathcal{D}(A^\triangleright)$.

*An analytic ring* $A$ *is an uncomplete analytic ring* $A = (A^\triangleright, \mathcal{D}(A))$ *with* $A^\triangleright \in \mathcal{D}(A)$. *And in this case we will denote the left adjonit of the inclusion by* $A \otimes_{A^\triangleright} -$.

**Definition 2.2.2.** *A morphism $f\colon A \to B$ between analytic rings is a morphism of underlying condensed rings such that the forgetful functor $\mathcal{D}(B^{\triangleright}) \to \mathcal{D}(A^{\triangleright})$ sends $\mathcal{D}(B)$ to $\mathcal{D}(A)$.*

**Example 2.2.3.** 1. *The inital analytic ring is $\mathbb{Z} = (\mathbb{Z}, \mathcal{D}(\mathsf{CondAb}))$, the ring of condensed integers. More generally, for any condensed ring $A$, there is a trivial analytic ring structure given by $A = (A, \mathcal{D}(A))$.*

2. *The solid integers $\mathbb{Z}^{\blacksquare} = (\mathbb{Z}, \mathcal{D}(\mathsf{Solid}))$ is also an analytic ring.*

The category of condensed ring can be a natural subcategory of AnRing via the trivial analytic ring structure.

**Proposition 2.2.4.** *The natural functor $F\colon \mathsf{CondRing} \to \mathsf{AnRing}^{u}$ is fully faithful. In particular, its right adjoint maps an uncomplete analytic ring to its underlying condensed ring.*

**Proposition 2.2.5.** *For an analytic ring $A = (A^{\triangleright}, \mathcal{D}(A))$, there exists a unique symmetric monoidal structure $\otimes_A$ on $\mathcal{D}(A)$ such that the functor $A \otimes_{A^{\triangleright}} -$ is symmetric monoidal.*

*Moreover, given a morphism $A \to B$ between analytic rings, the functor*

$$\mathcal{D}(A^{\triangleright}) \xrightarrow{B^{\triangleright} \otimes_{A^{\triangleright}} -} \mathcal{D}(B^{\triangleright}) \xrightarrow{B \otimes_{B^{\triangleright}}} \mathcal{D}(B)$$

*factors (uniquely) through*

$$\mathcal{D}(A^{\triangleright}) \xrightarrow{A \otimes_{A^{\triangleright}} -} \mathcal{D}(A) \xrightarrow{B \otimes_A -} \mathcal{D}(B),$$

*where the functor $B \otimes_A -$ is the left adjonit of the forgetful functor $\mathcal{D}(B) \to \mathcal{D}(A)$.*

*Proof.* (1): It suffices to show that the kernel of the completion functor is a $\otimes$-ideal. Let $M, C \in \mathcal{D}(A^{\triangleright})$ with $M \otimes_{A^{\triangleright}} A = 0$, then for any $N \in \mathcal{D}(A)$, we have

$$\begin{aligned} R\underline{\mathrm{Hom}}_A(A \otimes_{A^{\triangleright}} (C \otimes_{A^{\triangleright}} M), N) &= R\underline{\mathrm{Hom}}_{A^{\triangleright}}(C \otimes_{A^{\triangleright}} M, N) \\ &= R\underline{\mathrm{Hom}}_{A^{\triangleright}}(M, R\underline{\mathrm{Hom}}_{A^{\triangleright}}(C, N)) \\ &= R\underline{\mathrm{Hom}}_A(A \otimes_{A^{\triangleright}} M, R\underline{\mathrm{Hom}}_{A^{\triangleright}}(C, N)) = 0, \end{aligned}$$

which gives desired result.

(2): It suffices to show that this functor kills the kernel of $A \otimes_{A^\triangleright} -$, let $M \in \mathcal{D}(A^\triangleright)$ scuh that $A \otimes_{A^\triangleright} M = 0$ and let $K \in \mathcal{D}(B)$, then we have

$$\begin{aligned} R\underline{\mathrm{Hom}}_B(B \otimes_{B^\triangleright} (B^\triangleright \otimes_{A^\triangleright} M), K) &= R\underline{\mathrm{Hom}}_{B^\triangleright}(B^\triangleright \otimes_{A^\triangleright} M, K) \\ &= R\underline{\mathrm{Hom}}_{A^\triangleright}(M, K) \\ &= R\underline{\mathrm{Hom}}_A(A \otimes_{A^\triangleright} M, K) = 0. \end{aligned}$$

□