Derived Categories and Geometric Invariant Theory

Lecture Notes

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Contents

1 Letcure 2 **2**
- 1.1 A quick tour . . . 2
  - 1.1.1 Triangulated structure . . . 3
  - 1.1.2 Stable ∞-category . . . 5
  - 1.1.3 Derived category and Derived functors . . . 7
- 1.2 Coherent Sheaves . . . 8
- 1.3 Fourier-Mukai transform . . . 9

2 Lecture 3 **14**
- 2.1 Semiorthogonal decomposition . . . 14
- 2.2 Beilinson resolution . . . 15
- 2.3 Mutation of exceptional sequence . . . 20
- 2.4 More example . . . 21
  - 2.4.1 Derived category of the Grassmann . . . 21
  - 2.4.2 Kuznetsov component . . . 22

3 Lecture 4 **23**
- 3.1 Quotient stack . . . 23
  - 3.1.1 Naive example . . . 23
  - 3.1.2 Stack and group action . . . 24
  - 3.1.3 Quasi-coherent sheaves on stack . . . 26

3.2 Derived equivalence and birational geometry 28
3.2.1 Standard flip 29
3.2.2 Mukai flop 31

1 Letcure 2

1.1 A quick tour

The derived category is originally a technical tool to generalizing the Serre duality to the relative setting, now the derived category has become useful tools and interesting objects in many fields.

Definition 1.1. A morphism of complexes $f\colon A^\bullet \to B^\bullet$ is a quasi-isomorphism if the induced map $H^i(f)\colon H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all $i \in \mathbb{Z}$.

The central idea for the definition of the derived category is this: we should not distinguish between two quasi-isomorphism complexes. Example: For some space we can define many different cohomology theory but they have same cohomology groups, and only considering the cohomology group is not enough.

Definition 1.2. For an abelian category $\mathcal{A}$, its derived category is defined as

$$D(\mathcal{A}) := \mathrm{Ch}(\mathcal{A})[\mathrm{qis}^{-1}].$$

Roughly speaking, the objects in $D(\mathcal{A})$ is given by chain complex in $\mathcal{A}$, and the morphism between two complexes A, B is given by the diagram $A \to I \leftarrow B$, where I is an injective resolution of B with a homotopy class of the map $A \to I$. (We always assume that $\mathcal{A}$ has enough injectives and all complex is bounded.)

It is not trivial to construct the derived category, and we need to understand the localization functor $\mathrm{Ch}(\mathcal{A}) \to D(\mathcal{A})$ to help us handle the derived category. And there is a serious problem that contrary to $\mathrm{Ch}(\mathcal{A})$ is abelian, the derived category $D(\mathcal{A})$ is in general not abelian, but it still admits a triangulated structure. We will introduce the definition of triangulated category first, then we will compare it will the stable ∞-category.

We assume that reader are familiar with some basic category theory like abelian category. A good introduction is [Sch22]. The following is mainly follow with [Huy06] and [Li23].

1.1.1 Triangulated structure

Traditionally the derived category $D(\mathcal{A})$ is constructed as the localization of the homotopy category $K(\mathcal{A})$ with triangulated structure. Since $D(\mathcal{A})$ is never abelian unless all short exact sequence in $\mathcal{A}$ splits, we need the triangulated structure to get "homotopy limit" and "homotopy colimit".

Definition 1.3. Let $\mathcal{D}$ be an additive category with a functor $[1] : \mathcal{D} \to \mathcal{D}$ which gives an equivalence.

We define a triangle to be a sextuple (X, Y, Z, f, g, h) where $X, Y, Z \in Ob(\mathcal{D})$ and $f : X \to Y, g : Y \to Z, h : Z \to X[1]$ are morphisms of $\mathcal{D}$. There are natural morphism between triangles:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

Definition 1.4. A triangulated category is an triple $(\mathcal{D}, [1], \mathcal{H})$ where $\mathcal{H}$ is a set of triangles called the distinguished triangles. And $\mathcal{H}$ satisfies the following conditions:

(TR0) Any triangle isomorphic to a distinguished triangle is a distinguished triangle.

(TR1) Any triangle of the form $(X, X, 0, id, 0, 0)$ is distinguished.

(TR2) For any morphism $f : X \to Y$ there exists a distinguished triangle of the form (X, Y, Z, f, g, h).

(TR3) The triangle (X, Y, Z, f, g, h) is distinguished if and only if the triangle $(Y, Z, X[1], g, h, -f[1])$ is.

(TR4) Given a commutaitive diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \vdots{\scriptstyle c} & & \downarrow{\scriptstyle a[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

there exists a morphism $c : Z \mapsto Z'$ such that (a, b, c) is a morphism of triangles.

(TR5) Given distinguished triangles:

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \longrightarrow X[1]$$
$$Y \xrightarrow{g} Z \xrightarrow{k} X' \longrightarrow Y[1]$$
$$X \xrightarrow{gf} Z \xrightarrow{m} Y' \longrightarrow X[1]$$

there exists an distinguished triangle (Z', Y', X', u, v, w) such that the following diagram commutes:

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{h} & Z' & \longrightarrow & X[1] \\
\| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle u} & & \| \\
X & \xrightarrow{gf} & Z & \xrightarrow{m} & Y' & \longrightarrow & X[1] \\
\downarrow{\scriptstyle f} & & \| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle f[1]} \\
Y & \xrightarrow{g} & Z & \xrightarrow{k} & X' & \longrightarrow & Y[1] \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle m} & & \| & & \downarrow{\scriptstyle h[1]} \\
Z' & \xrightarrow{u} & Y' & \xrightarrow{v} & X' & \xrightarrow{w} & Z'[1]
\end{array}$$

We will call $\mathcal{D}$ is a pre-triangulated category if only TR0-TR4 hold.

TR5 is called the octahedron axiom, it seems very strange and complicatied. We give an example to inform the reader it is natural. And we should note the octahedral axiom is an easy consequence of ∞-categorical principles ([1.1.1.24 Lur17]), and the definition of a stable ∞-category is much simpler than a triangulated category.

Example 1.5. For a R-module sequences:

$$A \xrightarrow{f} B \xrightarrow{g} C$$

We have an exact sequence:

$$\begin{aligned} 0 \to \ker(f) \to \ker(g \circ f) \to \ker(g) \\ \to \operatorname{coker}(f) \to \operatorname{coker}(g \circ f) \to \operatorname{coker}(g) \to 0. \end{aligned}$$

We may consider this example in the homotopy category of complexes $\mathsf{K}(R)$, then use TR2, TR3 and TR5 to show this.

There are some natural definitions and simple propositions, we omit the proof of most of them.

Definition 1.6. The triangulated functor between triangulated categories is a functor F with given isomorphism between $F \circ [1]$ and $[1] \circ F$ such that F maps distinguished triangles into distinguished triangles.

Definition 1.7. Let $\mathcal{D}$ be a pre-triangulated category and $\mathcal{A}$ be an abelian category. An additive functor $H : \mathcal{D} \to \mathcal{A}$ is called cohomological if for every distinguished triangle (X, Y, Z, f, g, h) the sequence $HX \to HY \to HZ$ is exact.

Proposition 1.8. *Let $\mathcal{D}$ be an pre-triangulated category and $S \in \mathcal{D}$, then functor $\mathrm{Hom}(S, \cdot)$ and $\mathrm{Hom}(\cdot, S)$ are cohomological.*

And for a triangulated category $\mathcal{C}$ with a t-structure, the functor $\mathrm{H}^0 : \mathcal{C} \to \mathcal{C}^\heartsuit$ defined by $\mathrm{H}^0 = \tau_{\geq 0}\tau_{\leq 0} = \tau_{\leq 0}\tau_{\geq 0}$ is a cohomological functor.

Definition 1.9 (Mapping cone)**.** For a chain map $f \colon X \to Y \in \mathrm{Ch}(\mathcal{A})$, its mapping cone is defined as following:

$$\mathrm{cone}(f) := (X[1] \oplus Y, \begin{pmatrix} d_{X[1]} & 0 \\ f[1] & d_Y \end{pmatrix})$$

Theorem 1.10. *The derived category $D(\mathcal{A})$ has a triangulated structure, where the distinguished triangles is given by the triangles of the form*

$$X \xrightarrow{f} Y \to \mathrm{cone}(f) \to X[1].$$

1.1.2 Stable ∞-category

Let us recall what is a stable ∞-category.

Definition 1.11. An ∞-category $\mathcal{C}$ is stable if is satisfies the following conditions:

(1) There exists a zero object $0 \in \mathcal{C}$.

(2) Every morphism in $\mathcal{C}$ admits a fiber and a cofiber.

(3) A triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence.

The essential examples are given by the ∞-category Sp of the category of spectra and the derived ∞-category $\mathrm{D}(\mathcal{A})$ for suitable abelian category $\mathcal{A}$.

Theorem 1.12 (Schwede-Shipley)**.** *For a ring R, the derived category $D(R)$ is a stable ∞-category with all colimits, generated by a distinguished compact object 1 with an identification $\pi_0 \operatorname{Map}(1,1) = R^{op}$, such that $\pi_0 \operatorname{Map}(\Sigma^d 1, 1) = 0$ for $d \neq 0$.*

Example 1.13 (Suspension in derived category)**.** The cofiber of the zero map $X \to 0$ is given by $\Sigma X = X[1]$.

Example 1.14 (Cofiber in derived category)**.** The cofiber of a chian map $f\colon X \to Y$ is given by $\operatorname{cone}(f)$.

Now we give a triangulated category structure on stable ∞-category.

Definition 1.15. Let $\mathcal{C}$ be a pointed ∞-category which admits cofibers. Suppose given a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in the homotopy category $h\mathcal{C}$. We say this diagram is a distinguished triangle if there exists a diagram $\Delta^1 \times \Delta^2 \to \mathcal{C}$:

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \tilde{g}} & & \downarrow \\ 0' & \longrightarrow & Z & \xrightarrow{\tilde{h}} & W \end{array}$$

satisfying the following conditions:

Both squares are pushout diagrams in $\mathcal{C}$, and $h : Z \to X[1]$ is the composition of $\tilde{h}$ with the equivalence $W \simeq X[1]$ determined by the outer rectangle.

Theorem 1.16. *Let $\mathcal{C}$ be a pointed ∞-category which admits cofibers, and suppose the suspension functor Σ is an equivalence. Then $h\mathcal{C}$ has the structure of a triangulated category which is given by the above definition.*

Proof. [1.1.2.14 Lur17] □

Theorem 1.17. *The homotopy category of a stable ∞-category admits a caonical triangulated structure.*

Remark 1.18. Let $\mathcal{C}$ be a pointed ∞-category which admits cofiber, then the cofiber of a morphism is uniquely determined up to equivalence. Consider the full subcategory $\mathcal{E} \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by the cofiber sequences. Let $\theta : \mathcal{E} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ be the forgetful functor, which associateds to a diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

the morphism $g : X \to Y$. Applying [4.3.2.15 Lur09] twice, we see that θ is a trivial Kan fibration, therefore admits section. So the functor $\mathrm{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \to \mathcal{C}$, where the first map is given by section and the second is given by evaluation at the final object, is well-defined up to equivalence. Then we see the triangulated category in 1.16 has functorial cofiber.

And we can define the homology to get the usual long exact sequence.

Definition 1.19. For $d \in \mathbb{Z}$, define the functor $H_d \colon D(R) \to \mathrm{Mod}_R$ by

$$H_d(X) = [1, X]_d = [\Sigma^d X, Y] = \pi_0 \mathrm{Map}(\Sigma^d X, Y).$$

1.1.3 Derived category and Derived functors

Let $F \colon \mathcal{A} \to \mathcal{B}$ be an additive functor between adelian category, if F is an exact functor, then it maps quasi-isomorphism to the quasi-isomorphism, thus there is an extension functor $F \colon D(\mathcal{A}) \to D(\mathcal{B})$. In general such extension does not exist, so we need to find its universal approxiamtion.

Definition 1.20.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ \downarrow & & \downarrow \\ D(\mathcal{A}) & \dashrightarrow & D(\mathcal{B}) \end{array}$$

For an additive functor $F \colon \mathcal{A} \to \mathcal{B}$, if the left Kan extension exists and is exact, then it is denoted by RF, the right derived functor. If the right Kan extension exists and is exact, then its is denoted by LF, the left derived functor. If the derived functor exists, then we set

$$R^n F := H^n \circ RF, L_n F := H^{-n} \circ LF.$$

By definition, there exists long exact sequence of derived functors.

Theorem 1.21. *If the ableian category $\mathcal{A}$ have enough injectives (or projectives), then the right (left) derived functor exists.*

Proof. Assume that $\mathcal{A}$ has enough injectives, let $I_{\mathcal{A}}$ be the full subcategory of $\mathcal{A}$, then the natural functor $K^+(I_{\mathcal{A}}) \to D^+(\mathcal{A})$ is an equivalence, so we can define the extension via injective resolution. □

In general, we will only consider the right derived functor of left exact functor and left derived functor of right exact functor.

Proposition 1.22. *Let $\mathcal{C}$ be an abelian category with enough injectives, then there are natural isomorphisms*

$$\mathrm{Ext}^i(A, B) \simeq \mathrm{Hom}_{D(\mathcal{C})}(A, B[i]),$$

where $A, B \in \mathcal{C}$, $i \in \mathbb{Z}$.

Proof. FIx an injective resolution $B \simeq I \in D(\mathcal{C})$, then by construction we have

$$\mathrm{Ext}^i(A, B) \simeq H^i(\mathrm{Hom}(A, I)) \simeq \mathrm{Hom}(A, I[i]).$$

□

1.2 Coherent Sheaves

We always assume X is a projective variety over a field k, and all functor should be understand as its derived version. The derived category of X is definied as $D(X) := D^b(\mathrm{Coh}(X))$. In general the abelian category $\mathrm{Coh}(X)$ contains no non-trivial injective objects, so we need to work with the category of quasi-coherent sheaves to compute derived functors.

Proposition 1.23. *Let X be a noetherian scheme, then the natural functor $D(X) \to D^b(\mathrm{Qcoh}(X))$ identifies $D(X)$ with the subcategory $D^b_{Coh}(\mathrm{Qcoh}(X))$ of bounded complexes of quasi-coherent sheaves with coherent cohomology.*

Now let ω_X be the canonical bundle of the scheme X, the classical Serre duality tells us that for an coherent sheaf $\mathcal{F}$ on X, there is a natural isomorphsim $Ext^i(\mathcal{F}, \omega_X) \simeq H^{n-i}(X, \mathcal{F})^\vee$.

Definition 1.24. Let $\mathcal{A}$ be a k-linear triangulated category, then a Serre functor is an exact functor $S\colon \mathcal{A} \to \mathcal{A}$ with natural isomorphisms $\mathrm{Hom}(A,B) \simeq \mathrm{Hom}(B, S(A))^\vee$ for any objects $A, B \in \mathcal{A}$.

Theorem 1.25 (Serre duality)**.** *Let X be a smooth projective variety over a field k, the the functor $\omega_X \otimes -[n]$ is a Serre functor.*

Proof. This theorem says that for any two complexes $\mathcal{E}, \mathcal{F} \in D(X)$, there exists a functorial isomorphsim

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{E} \otimes \omega_X[n])^\vee.$$

By $\mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) \simeq \mathrm{Hom}(\mathcal{E}, \mathcal{F}[i])$, the above ismorphism can be stated as

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) \simeq \mathrm{Ext}^{n-i}(\mathcal{F}, \mathcal{E} \otimes \omega_X)^\vee,$$

which is a slight modification of the standard Serre duality. □

Remark 1.26. The Serre commutes with any equivalence, and it is unique up to unique isomorphism. If we have an equivalence $D(X) \simeq D(Y)$ for some smooth projective varieties X, Y, we should expect that they have "same" dimension and canonical bundle.

Theorem 1.27 (Gabriel,Rosenberg)**.** *Let X, Y be quasi-separated schemes, if we have an equivalence $\mathrm{Qcoh}(X) \simeq \mathrm{Qcoh}(Y)$, then $X \simeq Y$.*

Theorem 1.28 (Bondal-Orlov)**.** *Let X, Y be smooth projective varieties and suppose that the canonical bundle ω_X of X is ample or anti-ample. If there is an equivalence $D^b(X) \simeq D^b(Y)$, then we get $X \simeq Y$.*

1.3 Fourier-Mukai transform

Let X and Y be smooth projective varieties and denote the two projections by

$$q\colon X \times Y \to X, p\colon X \times Y \to Y.$$

Definition 1.29. Let $\mathcal{P} \in D^b(X \times Y)$, we call it as the Fourier-Mukai kernel, and its associated Fourier-Mukai transform is the functor

$$\begin{aligned} \Phi_{\mathcal{P}}\colon D^b(X) &\to D^b(Y) \\ \mathcal{E}^\bullet &\mapsto p_*(q^*\mathcal{E}^\bullet \otimes \mathcal{P}). \end{aligned}$$

Example 1.30. The identity functor $\mathrm{id}\colon D(X) \to D(X)$ is naturally isomorphic to the Fourier-Mukai transform $\Phi_{\mathcal{O}_\Delta}$, where $\Delta \subset X \times X$ is the diagonal. Indeed, with $\tau\colon X \simeq \Delta$, we have

$$\begin{aligned}\Phi_{\mathcal{O}_\Delta}(\mathcal{E}) &= p_*(q^*\mathcal{E} \otimes \mathcal{O}_\Delta) = p_*(q^*\mathcal{E} \otimes \tau_*\mathcal{O}_X)\\ &\simeq p_*(\tau_*(\tau^*q^*\mathcal{E} \otimes \mathcal{O}_X))\\ &\simeq (p\circ\tau)_*(q\circ\tau)^*\mathcal{E} \simeq \mathcal{E}.\end{aligned}$$

Example 1.31. By the same calculation, for a map $f\colon X \to Y$, the induced map $f_*\colon D(X) \to D(Y)$ is naturally isomorphic to the Fourier-Mukai transform $\Phi_{\mathcal{O}_{\Gamma_f}}$, where $\Gamma_f \subset X \times Y$ is the graph of f. For the convese induced map $f^*\colon D(Y) \to D(X)$, it also isomorphic to the Fourier-Mukai transform with kernel $\mathcal{O}_{\Gamma_f}$.

Example 1.32. The Serre functor can be identifies with the Fourier-Mukai transform with kernel $\tau_*\omega_X[n]$.

Example 1.33. Let $\mathcal{P} \in D(X \times Y)$ be a coherent sheaf flat over X, for a k-rational point $x \in X$(i.e. a closed point $x \in X$ with $k(x) \simeq k$), we have

$$\begin{aligned}\Phi_{\mathcal{P}}(k(x)) &= p_*(q^*k(x) \otimes \mathcal{P}) = p_*(i_*i^*\mathcal{O}_{X\times Y} \otimes \mathcal{P})\\ &= p_*i_*(i^*\mathcal{O}_{X\times Y} \otimes i^*\mathcal{P})\\ &= p_*(i_*i^*\mathcal{P}) = \mathcal{P}|_{x\times Y},\end{aligned}$$

where i is the closed immersion $x \times Y \to X \times Y$.

The Fourier-Mukai transforms have many fine properties. The composition of Fourier-Mukai transforms is still Fourier-Mukai transform, the Fourier-Mukai transform has both left adjoint and right adjoint which are Fourier-Mukai transforms, too. And both these transforms have kernel which can be described explicitly. The following celebrated theorem tell us that the Fourier-Mukai transforms are the essential functor we should care about in geometric case.

Theorem 1.34. *Let X and Y be smooth projective varieties and assume we have a fully faithful exact functor*

$$F\colon D^b(X) \to D^b(Y).$$

Then there exists a unique object $\mathcal{P} \in D^b(X \times Y)$ up to isomorphism such that $F \simeq \Phi_{\mathcal{P}}$.

The proof of this theorem is not easy, but with ∞-category theory we can reformulate it and easily prove it.

Theorem 1.35 ([BFN10])**.** *For any quasi-compact, quasi-separated k-scheme X, Y, there is a canonical equivalence of stable ∞-categories*

$$\mathrm{Fun}^L(D(X), D(Y)) \simeq D(X \times Y).$$

In [Toë07], Toën also identifies the dg category of k-linear continuous (i.e. colimit preserving) functors with the dg category of integral kernels

$$\mathrm{Fun}_k(\mathrm{Qcoh}(X), \mathrm{Qcoh}(Y)) \simeq \mathrm{Qcoh}(X \times_k Y)$$

for ordinary schemes over a ring k.

Definition 1.36 (Spanning class)**.** A collection Ω of objects is called a spanning class of $\mathcal{C}$ if for all $B \in \mathcal{C}$ the following conditions hold:

1. If $\mathrm{Hom}(A, B[i]) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \simeq 0$.
2. If $\mathrm{Hom}(B[i], A) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \simeq 0$.

If $\mathcal{C}$ admits a Serre functor, then the two conditions are equivalent.

Proposition 1.37. *Let X be a smooth projective variety, then the objects of form $k(x)$ with $x \in X$ a closed point span the whole catgeory $D(X)$.*

Proof. It suffices to show that for any non-trivial complex $\mathcal{F} \in D(X)$, there exists closed points x such that

$$\mathrm{Hom}(\mathcal{F}, k(x)[i]) \neq 0$$

for integers i.

Consider the spectral sequence

$$E_2^{p,q} = \mathrm{Hom}(H^{-q}(\mathcal{F}), k(x)[p]) \Rightarrow \mathrm{Hom}(\mathcal{F}, k(x)[p+q]),$$

take maximal m such that $H^m(\mathcal{F}) \neq 0$, then $E_\infty^{0,-m} = E_2^{0.-m}$. If we choose a closed point x in the support of $H^m(\mathcal{F})$, then $\mathrm{Hom}(H^m(\mathcal{F}), k(x))$ is non-trivial, hence $\mathrm{Hom}(\mathcal{F}, k(x)[-m]) \neq 0$. □

Proposition 1.38. *For an exact functor $F\colon \mathcal{C} \to \mathcal{D}$ with left and right adjonits, then if $\mathrm{Hom}(A, B[i]) \to \mathrm{Hom}(F(A), F(B)[i])$ is bijective for all $A, B \in \Omega$, where Ω is a spanning class of $\mathcal{C}$, then F is fully faithful.*

Theorem 1.39 (Bondal,Orlov)**.** *Over a algebraiclly closed field k of characteristic 0, a Fourier-Mukai functor $\Phi\colon D(X) \to D(Y)$ is fully faithful if and only if for any two closed points $x, y \in X$ we have*

$$\mathrm{Hom}(\Phi(k(x)), \Phi(k(x))) = k$$
$$\mathrm{Hom}(\Phi(k(x)), \Phi(k(y))[i]) = 0 \text{ if } x \neq y \text{ or } i < 0 \text{ or } i > \dim x.$$

Proof. The proof is an application of above proposition. Since we know that objects of the form $k(x)[i]$ with $x \in X$ a closed point form a spanning class in $D^b(X)$, the functor F is fully faithful if and only if the following map

$$\mathrm{Hom}(k(x), k(y)[i]) \to \mathrm{Hom}(F(k(x)), F(k(y))[i])$$

is an isomorphism for any closed points $x, y \in X$ and any integer i. For $x \neq y$, this trivially holds by assumption. By adjunction, we have

$$\mathrm{Hom}(F(k(x)), F(k(x))[i]) \simeq \mathrm{Hom}(G(F(k(x))), k(x)[i]).$$

If we can show that $GF(k(x)) \simeq k(x)$, then the adjunction morphism $g\colon GF(k(x)) \to k(x)$ is either an isomorphism or zero. And clearly g is nonzero,since by the triangle equality we have $F(k(x)) \to FGF(k(x))$ is the identity.

We will show that $GF(k(x)) \simeq k(x)$ under two assumptions:

1. $GF(k(x))$ is a sheaf,
2. The map $\mathrm{Hom}(k(x), k(x)[i]) \to \mathrm{Hom}(GF(k(x)), k(x)[i])$ is injective for $i = 1$.

Set $\mathcal{F} = GF(k(x))$, one has $Hom(\mathcal{F}, k(y)) = \mathrm{Hom}(F(k(x)), F(k(x))) = 0$ for any closed point $y \neq x$, so the sheaf $\mathcal{F}$ is concentrated at x. As before, the adjunction map $g\colon \mathcal{F} \to k(x)$ is nontrival hence surjective. We need to show that g is in fact bejective. Consider the exact sequence

$$0 \to \ker(g) \to \mathcal{F} \to k(x) \to 0,$$

after applying the functor $\mathrm{Hom}(-.k(x))$, we get

$$0 \to \mathrm{Hom}(\ker(g), k(x)) \to \mathrm{Hom}(k(x), k(x)[1]) \to \mathrm{Hom}(\mathcal{F}, k(x)[1])$$

since $\mathrm{Hom}(\mathcal{F}, k(x)) = k$, so the last map is injective hence $\ker(g) = 0$.

Now let us return to the proof of additional assumptions:

1.

Lemma 1.40. *Let X be a smooth projective variety and we fix a closed point $x \in X$. For $\mathcal{F} \in D(X)$, if $\mathrm{Hom}(\mathcal{F}, k(y)[i]) = 0$ for any closed point $y \neq x$ and $\mathrm{Hom}(\mathcal{F}, k(x)[i]) = 0$ for $i < 0, i > \dim X$, then $\mathcal{F}$ is isomorphic to a sheaf concentrated at x.*

Proof. We can consider the spectral sequence

$$E_2^{p,q} := \mathrm{Hom}(H^{-q}(\mathcal{F}), k(y)[p]) \Rightarrow \mathrm{Hom}(\mathcal{F}, k(y)[p+q]).$$

Let m_0 be maximal with $y \in \mathrm{supp}(H^{m_0}\mathcal{F})$, then $E_2^{0,-m_0} \neq 0$ and $E_2^{p,q} = 0$ for $q \leq -m_0$. Hence

$$0 \neq E_2^{0,-m_0} = E_\infty^{0,-m_0} = \mathrm{Hom}(\mathcal{F}, k(y)[-m_0]),$$

which implies $y = x$ and $-d \leq m_0 \leq 0$, where $d = \dim X$. We get that all cohomology sheaves of $\mathcal{F}$ are concentrated at x and in degree $-d \leq i \leq i$.

Now let m_1 be minimal with $H^{m_1}\mathcal{F} \neq 0$, by Serre duality we get

$$\mathrm{Hom}(H^{m_1}\mathcal{F}, k(x)[d]) \simeq \mathrm{Hom}(k(x), H^{m_1}\mathcal{F})^\vee \neq 0.$$

Then this term causes that $\mathrm{Hom}(\mathcal{F}, k(x)[d - m_1])$ is nonzero, so we get $m_1 = m_0 = 0$. □

2. Need some non-trivial result so we skip it.

□

Corollary 1.41. *Let $\mathcal{P}$ be a coherent sheaf on $X \times Y$ flat over X. Then $\Phi_\mathcal{P}$ is fully faithful if and only if the following two conditions hold:*

1. *For any closed point $x \in X$ one has $\mathrm{Hom}(\mathcal{P}_x, \mathcal{P}_x) \simeq k$.*
2. *If $x \neq y$, then $\mathrm{Ext}^i(\mathcal{P}_x, \mathcal{P}_y) = 0$ for all i.*

2 Lecture 3

2.1 Semiorthogonal decomposition

Definition 2.1. A full triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$ is called right admissible if the inclusion functor admits a right adjoint.

The (right) orthogonal complement of a subcategory $\mathcal{D}' \subset \mathcal{D}$ is the full subcategory $\mathcal{D}'^{\perp}$ spanned by objects $C \in \mathcal{D}$ such that $\mathrm{Hom}(B, C) = 0$ for all $B \in \mathcal{D}'$.

Definition 2.2. A sequence of full right admissible triangulated subcategories

$$\mathcal{D}_1, \cdots, \mathcal{D}_n \subset \mathcal{D}$$

is semi-orthogonal if for all $i > j$ we have $\mathcal{D}_j \subset \mathcal{D}_i^{\perp}$.

Such a sequence defines a semi-orthogonal decomposition of $\mathcal{D}$ if $\mathcal{D}$ is generated by the $\mathcal{D}_i$. For a semi-orthogonal decomposition, we will denote it by

$$\mathcal{D} = \langle \mathcal{D}_1, \cdots, \mathcal{D}_n \rangle.$$

Remark 2.3. Why don't we consider the orthogonal decomposition of $\mathcal{D}$, which is given by two subcategories $\mathcal{D}_1, \mathcal{D}_2$ generated the whole category such that $\mathrm{Hom}(\mathcal{D}_1, \mathcal{D}_2) = \mathrm{Hom}(\mathcal{D}_2, \mathcal{D}_1) = 0$. If a variety X is the disjoint union of two varieties Y, Z, then there is an orthogonal decomposition of $D(X)$ given by $D(Y), D(Z)$. Under some mild conditions, we can show $D(X)$ admits an orthogonal decomposition if and only if X is not disconnected, so this kind of decomposition is not interesting.

Definition 2.4. An object $E \in \mathcal{C}$ is called exceptional if

$$R\,\mathrm{Hom}(E, E) \simeq k.$$

An exceptional sequence is a sequence $E_1, \cdots, E_n$ of exceptional objects such that $R\,\mathrm{Hom}(E_i, E_j) = 0$ for all $i > j$. An exceptional sequence is full if $\mathcal{C}$ is generated by $\{E_i\}$.

Example 2.5. Let $E_1, \cdots, E_n$ be an exceptional sequence in $\mathcal{D}$, then the admissible triangulated subcategories $\mathcal{D}_i := \langle E_i \rangle$ form a semi-orthogonal sequence. If this exceptional sequence is full, then the subcategories $\mathcal{D}_i$ form a semi-orthogonal decomposition.

Example 2.6. For any line bundle $\mathcal{O}(i), \mathcal{O}(j)$ o projective space $\mathbb{P}^n$ with $a \leq j \leq i \leq a+n$, we have

$$\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)[l]) \simeq H^l(\mathbb{P}^n, \mathcal{O}(j-i)) = \begin{cases} k & \text{if } l = 0, i = j \\ 0 & \text{otherwise} \end{cases},$$

thus, $\mathcal{O}(a), \mathcal{O}(a+1), \cdots, \mathcal{O}(a+n)$ is an exceptional sequence. In particular, we have this semi-orthogonal decomposition

$$D(\mathbb{P}^n) = \langle \mathcal{O}(a), \mathcal{O}(a+1), \cdots, \mathcal{O}(a+n) \rangle$$

for any $a \in \mathbb{Z}$.

Example 2.7. Let X be a Fano variety of index r over a field k of characteristic 0, i.e. $-K_X \equiv mH$ for some ample Cartier divisor H. Then $H^s(X, \mathcal{O}_X(-iH)) = 0$ for $1 \leq i \leq r-1$ by Kodaira vanishing, so $\mathcal{O}_X((1-r)H), \cdots, \mathcal{O}_X(-H), \mathcal{O}_X$ is an exceptional sequence in $D(X)$.

The notation of exceptional sequence only works well for Fano varieties, it can be shown that the derived category of Calabi-Yau varieties only admits trivial semi-orthogonal decomposition.

Remark 2.8. The definition of semi-orthogonal decomposition lacks some crucial information. Consdier the following two categories: $D(* \sqcup *) = \langle D(*), D(*) \rangle$ and $D(\mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(1) \rangle$. The decomposition of $D(* \sqcup *)$ is completely orthogonal, so there are no maps between two components in either direction. The decomposition of $\mathbb{P}^1$ is not orthogonal since we have $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) = H^0(\mathbb{P}^1, \mathcal{O}(1)) \simeq k^2 \neq 0$.

What the notation is hiding is the data of the maps in opposite direction, which should be throught as some gluing data.

2.2 Beilinson resolution

The Fourier-Mukai transform associated to the structure sheaf of the diagonal $\mathcal{O}_\Delta$ in $\mathbb{P}^n \times \mathbb{P}^n$ is just the identity functor, but there exits a special resolution of $\mathcal{O}_\Delta$ that gives a rich structure.

In this section, let $\mathbb{P}^n = \mathrm{Proj}\,\mathrm{Sym}\, V^\vee$ be the projective space associated with an $(n+1)$-dimensional vector space V over a field k with a fixed basis $x_0, \cdots, x_n \in H^0(\mathbb{P}^n, \mathcal{O}(1))$. We will use the notation $\mathcal{F} \boxtimes \mathcal{G} := p^*\mathcal{F} \otimes q^*\mathcal{G}$.

Lemma 2.9 ([Bel79]). *There exists a natural locally free resolution of $\mathcal{O}_\Delta$ of the form*

$$0 \to \bigwedge^n (\mathcal{O}(-1) \boxtimes \Omega(1)) \to \bigwedge^{n-1} (\mathcal{O}(-1) \boxtimes \Omega(1)) \to \cdots$$
$$\cdots \to \mathcal{O}(-1) \boxtimes \Omega(1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0.$$

We will denote this locally free resolution by $L^\bullet$.

Proof. Consider the Euler sequence twisted by $\mathcal{O}(1)$:

$$0 \to \Omega^1(1) \to V^\vee \otimes \mathcal{O} \to \mathcal{O}(1) \to 0,$$

this sequence at a point $l \in \mathbb{P}(V)$ is given by

$$0 \to l^\perp \to V^\vee \to l^\vee \to 0.$$

The homomorphism $\mathcal{O}(-1) \boxtimes \Omega(1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}$ as a point (l, l') is given by $(x, \varphi) \mapsto \varphi(x)$ where $x \in l$ and $\varphi|_{l'} = 0$. Clearly, the image of this map is the ideal sheaf of the diagonal Δ. By standard construction of the Koszul complex, we get the resolution.

We can give a detailed calculation in local coordinates. Consdier the twisted Euler sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{n+1} \to T(-1) \to 0,$$

we get an isomorphism $H^0(\mathbb{P}^n, T(-1)) \simeq H^0(\mathbb{P}^n, \mathcal{O}^{n+1}) = V$. A basis of V is given by $x_0^\vee, \cdots, x_n^\vee$, we denote its image in $H^0(\mathbb{P}^n, T(-1))$ by $\frac{\partial}{\partial x_i}$.

Consider the global section s of $\mathcal{O}(1) \boxtimes T(-1)$ given by

$$s = \sum x_i \otimes \frac{\partial}{\partial x_i} \in H^0(\mathcal{O}(1)) \otimes H^0(T(-1)) \simeq H^0(\mathcal{O}(1) \boxtimes T(-1)).$$

This section precisely gives us the desired map, in particular, the zero locus of s is just the diagonal. □

Theorem 2.10 (Beilinson spectral sequence). *For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ there exist two natural spectral sequence:*

$$E_1^{r,s} := H^s(\mathbb{P}^n, \mathcal{F}(r)) \otimes \Omega^{-r}(-r) \Rightarrow E_\infty^{r,s} = \begin{cases} \mathcal{F} & \text{if } r = s = 0, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$E_1^{r,s} := H^s(\mathbb{P}^n, \mathcal{F} \otimes \Omega^{-r}(-r)) \otimes \mathcal{O}(r) \Rightarrow E_\infty^{r,s} = \begin{cases} \mathcal{F} & \text{if } r = s = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider the spectral sequence

$$E_1^{r,s} = R^s F(A^r) \Rightarrow R^{r+s} F(A).$$

If we set $A := p^*\mathcal{F} \otimes L^\bullet$ and $F := q_*$, then

$$A^r = p^*\mathcal{F} \otimes (\mathcal{O}(-r) \boxtimes \Omega^{-r}(-r)) = \mathcal{F}(r) \boxtimes \Omega^{-r}(-r),$$

hence $R^s F(A^r) \simeq H^s(\mathbb{P}^n, \mathcal{F}(r)) \otimes \Omega^{-r}(-r)$ due to projection formula.

On the other hand, we have

$$A \simeq p^*\mathcal{F} \otimes L^\bullet \simeq p^*\mathcal{F} \otimes \mathcal{O}_\Delta \simeq \tau_*\mathcal{F},$$

where τ is the diagonal embedding. Therefore $q_*\tau_*\mathcal{F} \simeq \mathcal{F}$, this proves the statement.

By same argument with interchanging the role of p, q, we get another spectral sequence. □

Proposition 2.11. *Any sequence of lines bundles of the form*

$$\mathcal{O}(a), \mathcal{O}(a+1), \cdots, \mathcal{O}(a+n)$$

on $\mathbb{P}^n$ deines a full exceptional sequence in $D(\mathbb{P}^n)$.

Proof. It suffices to show this exceptional sequence is full, i.e any object $\mathcal{F}$ in the orthogonal complement $\langle \mathcal{O}(a), \cdots, \mathcal{O}(a+n) \rangle^\perp$ is trivial.

This result directly follows the Beilinson spectral sequence when $\mathcal{F}$ is a sheaf, applying the spectral sequence for the sheaf $\mathcal{F}$ with some twist, we get

$$E_1^{r,s} = H^s(\mathbb{P}^n, \mathcal{F}(-a)(r)) \otimes \Omega^{-r}(-r) \simeq \mathrm{Hom}(\mathcal{O}(i), \mathcal{F}[s]) \otimes \Omega^{-r}(-r),$$

where $i = a - r$. If $\mathcal{F}$ is orthogonal to any $\mathcal{O}(i)$ for $i = a, \cdots, a+n$, then all $E_1^{r,s}$ are trivial, hence the spectral sequence converges to $\mathcal{F}(-a) \simeq 0$, this shows that $\mathcal{F} \simeq 0$.

For general case, consider the spectral sequence

$$E_2^{p,q} = \mathrm{Hom}(\mathcal{O}(a), H^q(\mathcal{F}^\bullet)[p]) \Rightarrow \mathrm{Hom}(\mathcal{O}(a), \mathcal{F}^\bullet[p+q]).$$

Since the limit page is zero, the second page also vanishes, thus $H^q(\mathcal{F}^\bullet) = 0$ for all q.

There is another proof that does not require the spectral sequence, we shall split the resolution $L^\bullet \to \mathcal{O}_\Delta$ into short exact sequence:

$$0 \to M_k \to \bigwedge^{k-1} (\mathcal{O}(-1) \boxtimes \Omega(1)) \to M_{k-1} \to 0,$$

where $M_n = \bigwedge^{k-1}(\mathcal{O}(-1) \boxtimes \Omega(1)), M_0 = \mathcal{O}_\Delta$.

The short exact sequence can be regarded as the exact triangle in $D(\mathbb{P}^n \times \mathbb{P}^n)$, after taking tensor product with $q^*\mathcal{F}$ and direct image under the projection p, we get the exact triangle

$$\Phi_{M_{k+1}}\mathcal{F} \to \Phi_{\mathcal{O}(-i)\boxtimes\Omega^i(i)}\mathcal{F} \to \Phi_{M_i}\mathcal{F}.$$

Clearly, $\Phi_{\mathcal{O}(-i)\boxtimes\Omega^i(i)}\mathcal{F} \simeq R\Gamma(\mathbb{P}^n, \mathcal{F} \otimes \Omega^i(i)) \otimes \mathcal{O}(-i)$ is contained in $\langle \mathcal{O}(-i) \rangle$. Bu induction this shows that $\Phi_{M_i}\mathcal{F} \in \langle \mathcal{O}(-n), \cdots, \mathcal{O}(-i) \rangle$ for all i, and for $i = 0$, we get $\mathcal{F} \in \langle \mathcal{O}(-n), \cdots, \mathcal{O} \rangle$.

Thus, $\mathcal{O}(-n), \cdots, \mathcal{O}$ is a full exceptional sequence. Since tensor product with $\mathcal{O}(a+n)$ defines an equivalence, we get $\mathcal{O}(a), \cdots, \mathcal{O}(a+n)$ is also a full exceptional sequence. □

Remark 2.12. By same argument, we can show $\Omega^p(p)$ with $p = 0, \cdots, n$ form a full exceptional sequence in $D(\mathbb{P}^n)$.

Proposition 2.13 (Beilinson relation)**.** *For any pair i, j such that $0 \leq i, j \leq n$ and $l \geq 1$ we have*

$$\begin{aligned} R\,\mathrm{Hom}(\mathcal{O}(-i), \mathcal{O}(-j)) &= \mathrm{Sym}^{i-j}(V^\vee) \\ R\,\mathrm{Hom}(\Omega^i(i), \Omega^j(j)) &= \Lambda^{i-j}(V) \\ R\,\mathrm{Hom}(\mathcal{O}(i), \Omega^j(j)[j]) &= \delta_{ij}k. \end{aligned}$$

Proof. The first part comes from the cohomology of the line bundle on $\mathbb{P}^n$ and the identification $H^0(\mathbb{P}^n, \mathcal{O}(k)) = \mathrm{Sym}^k V^\vee$. The second part comes form the de Rham cohomology of the projective space $\mathbb{P}^n$ and induction on Euler sequence. □

Definition 2.14. An exceptional collection $E_1, \cdots, E_m \in D(X)$ is strong if in addition $\mathrm{Hom}(E_i, E_j[l]) = 0$ for all i, j and $l \neq 0$

Theorem 2.15 (Bondal)**.** *Let X be a smooth projective variety. Suppose there is a strong full exceptional sequence $E_1, \cdots, E_m \in D(X)$. Set $E = \oplus E_i$ and $A = \mathrm{End}(E)$, then*

$$R\,\mathrm{Hom}(E, -)\colon D(X) \to D(\text{mod-}A)$$

is an equivalence, where mod-A *is the category of finitely generated right A-module.*

Remark 2.16. This theorem gives us a new perspective on derived categories of varieties. In particular, there is a natural equivalence between the derived category of projective space and the derived category of a quiver. For the projective line, it is derived equivalent to the Kronecker quiver:

$$0 \underset{b}{\overset{a}{\rightrightarrows}} 1.$$

For general projective space, there can not exists quiver Q such that $D(\mathbb{P}^n) \simeq D(Q)$ due to dimesion, the solution is to impose a finite relation on the quiver. The Beilinson quiver

$$0 \underset{x_{n+1}^{(0)}}{\overset{x_1^{(0)}}{\rightrightarrows}} 1 \underset{x_{n+1}^{(1)}}{\overset{x_1^{(1)}}{\rightrightarrows}} 2 \rightrightarrows \cdots \underset{x_{n+1}^{(n-1)}}{\overset{x_1^{(n-1)}}{\rightrightarrows}} n$$

with relation $x_i^{(l+1)} x_j^{(l)} = x_j^{(l+1)} x_i^{(l)}$ for $0 \leq l \leq n-2$ and $1 \leq i, j \leq n+1$ is constructed to have the path algebra isomorphic to $\mathrm{End}(\oplus_0^n \mathcal{O}(a))$, and is derived equivalent to $\mathbb{P}^n$.

The result of the derived category of projective space can be easily generalized to the raltive setting. In particular, we have the following theorem.

Theorem 2.17 (Orlov)**.** *Let $\mathcal{E}$ be a vector bundle of rank r over a projective smooth variety X, we denote its associated projective bundle by $\pi\colon \mathbb{P}(\mathcal{E}) \to X$. Then for any $a \in \mathbb{Z}$ the sequence of full subcategories*

$$\pi^* D(X) \otimes \mathcal{O}(a), \cdots, \pi^* D(X) \otimes \mathcal{O}(a + r - 1)$$

deines a semi-orthogonal decomposition of $D(\mathbb{P}(\mathcal{E}))$.

Theorem 2.18 ([Ber09]). *Let $p\colon Y \to X$ be a Brauer-Severi variety (éatle locally $\mathbb{P}^n$-bundle) and let β be the corresponding Brauer class, then we have the semi-orthogonal decomposition*

$$D(Y) = \langle p^*D(X), p^*D(X,\beta^{-1}) \otimes \mathcal{O}(1), \cdots, p^*D(X,\beta^{-1}) \otimes \mathcal{O}(n)\rangle.$$

Theorem 2.19 (Orlov). *Let $Y \subset X$ be a smooth subvariety of codimension c, consider the following blow-up diagram*

$$\begin{array}{ccc} E & \xrightarrow{i} & \mathrm{Bl}_Y X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \longrightarrow & X \end{array} .$$

The map p is just the projectivization of the normal bundle of Y in X, in this case, we have the following decomposition

$$D(\mathrm{Bl}_Y X) = \langle \pi^*D(X), i_*p^*D(Y), i_*p^*D(Y)\otimes\mathcal{O}(1), \cdots, i_*p^*D(Y)\otimes\mathcal{O}(c-2)\rangle.$$

2.3 Mutation of exceptional sequence

For the derived category $D(X)$ of a smooth proejctive variety, every component of a semi-orthogonal decomposition is (left and right) admissible,i.e. its inclusion functor admits both left and right adjoints.

If we have an admissible subcategory $\mathcal{B} \subset \mathcal{D}$ with inclusion functor β and its left adjoint β^* and right adjoint $\beta^!$, it gives a semi-orthogonal decomposition $\mathcal{D} = \langle \mathcal{A}, \mathcal{B}\rangle$, we can also write as $\mathcal{D} = \langle \mathcal{B}, \mathcal{C}\rangle$, where $\mathcal{A} = \mathcal{B}^\perp$ is the right orthogonal complement of $\mathcal{B}$, $\mathcal{C} = {}^\perp\mathcal{B}$ is the left orthogonal complement of $\mathcal{B}$. There exists left and right mutation functors of $\mathcal{B}$, which is denoted as $L_\mathcal{B}$ and $R_\mathcal{B}$, they fit the folllowing triangles

$$\beta\beta^! \to \mathrm{id} \to L_\mathcal{B}, R_\mathcal{B} \to \mathrm{id} \to \beta\beta^*.$$

Proposition 2.20 ([BK90]). *If we have a semi-orthogonal decomposition*

$$\mathcal{D} = \langle \mathcal{A}_1, \cdots, \mathcal{A}_i, \cdots, \mathcal{A}_n\rangle,$$

then for any i, the following are semi-orthogonal decomposition:

$$\mathcal{D} = \langle \mathcal{A}_1, \cdots, \mathcal{A}_{i-1}, \mathcal{A}_{i+1}, R_{\mathcal{A}_{i+1}}\mathcal{A}_i, \mathcal{A}_{i+2}, \cdots, \mathcal{A}_n\rangle$$
$$\mathcal{D} = \langle \mathcal{A}_1, \cdots, \mathcal{A}_{i-1}, L_{\mathcal{A}_i}\mathcal{A}_{i+1}, \mathcal{A}_i, \mathcal{A}_{i+2}, \cdots, \mathcal{A}_n\rangle$$

Proposition 2.21 ([BK90]). *For any semi-orthogonal decomposition of length n, consider the mutation functor R_i, L_i as transformation on the set of collection of subcategories, we have*

1. $R_i = L_i^{-1}$,
2. $R_i \circ R_{i+1} \circ R_i = R_{i+1} \circ R_i \circ R_{i+1}$.

Example 2.22. For the naive case $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle = \langle \mathcal{B}, \mathcal{C} \rangle$, the mutation functor gives an equivalence between $\mathcal{A}$ and $\mathcal{C}$.

Example 2.23. Starting form the exceptional sequence in Beilinson theorem, we can get new exceptional sequences via mutation. For an exceptional object E, its left and right mutation functor is given by

$$R\operatorname{Hom}(E, -) \otimes E \xrightarrow{ev} \mathrm{id} \to L_E,$$
$$R_E \to \mathrm{id} \xrightarrow{coev} R\operatorname{Hom}(-, E)^\vee \otimes E.$$

In particular, the dual of the exceptional sequence $\langle \mathcal{O}, \cdots, \mathcal{O}(n) \rangle$ is given by the exceptional sequence $\langle \Omega^n(n)[n], \Omega^{n-1}(n-1)[n-1], \cdots, \Omega^1(1)[1], \mathcal{O} \rangle$.

2.4 More example

2.4.1 Derived category of the Grassmann

The derived category of Grassmann or flag varieties is kind of complicated [Fon24]. Let V be a vector space of dimension n over an algebraiclly clsoed field k of characteristic 0, then there is a clean result of derived category of Grassmann.

Theorem 2.24 ([Kap85]). *The bounded derived category of $\mathrm{Gr}(k, V)$ admits a full exceptional collection indexed by Young diagram*

$$D(\mathrm{Gr}(k, V)) = \langle \Sigma^\lambda \mathcal{U}^\vee \mid \lambda \in Y_{k,n-k} \rangle,$$

where $\mathcal{U}$ is the tautological bundle. The dual of this exceptional sequence is given by

$$D(\mathrm{Gr}(k, V)) = \langle \Sigma^\mu \mathcal{U}^\perp \mid \mu \in Y_{n-k,k} \rangle.$$

For $\lambda \in Y_{k,n-k}$ and $\mu \in Y_{n-k,k}$ one has

$$R\operatorname{Hom}(\Sigma^\lambda \mathcal{U}^\vee, \Sigma^\mu \mathcal{U}^\perp) = \begin{cases} k[-|\lambda|] & \text{if } \lambda = \mu^T, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.25. *Let $\mathcal{E}$ be a vector bundle of rank n on a smooth projective variety X, and let $1 \leq k \leq n-1$ be an integer. Consider the relative Grassmannian bundle $\pi : \mathrm{Gr}_X(k, \mathcal{E}) \to X$, and let $\mathcal{U} \subset \pi^*\mathcal{E}$ denote the relative tautological bundle. Then there is a (graded) semiorthogonal decomposition of the form*

$$D(\mathrm{Gr}_X(k, \mathcal{E})) = \left\langle \pi^* D(X) \otimes \Sigma^{\lambda} \mathcal{U}^{\vee} \mid \lambda \in Y_{k,n-k} \right\rangle.$$

In particular, if $D(X)$ has a full exceptional collection (consisting of vector bundles), then $D(\mathrm{Gr}_X(k, \mathcal{E}))$ also has a full exceptional collection (consisting of vector bundles).

Corollary 2.26. *Let X be a smooth projective variety, which can be obtained as an iterated Grassmannian bundle over a smooth projective variety S such that $D(S)$ admits a full exceptional collection. Then so does $D(X)$.*

Since for all integer sequences $1 \leq i_1 < i_2 < \cdots < i_s \leq n-1$ the partial flag variety $\mathrm{Flag}(i_1, \ldots, i_s; V)$ is an iterated Grassmannian bundle, say, over the Grassmannian $\mathrm{Gr}(i_1, V)$, we conclude the following.

Corollary 2.27. *All (partial) flag varieties over an algebraically closed field of characteristic zero admit full exceptional collections consisting of equivariant vector bundles.*

2.4.2 Kuznetsov component

Consider a hypersurface $X \subset \mathbb{P}^N$ of degree $d \leq N$, by adjunction formula we get $\omega_X \simeq \mathcal{O}_X(-N-1+d)$, so X is a Fano variety, and there is a decomposition

$$D(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \cdots, \mathcal{O}_X(N-d) \rangle,$$

where $\mathcal{A}_X$ is (by definition) the orthogonal complement of the category spanned by $\mathcal{O}_X(i), 0 \leq i \leq N-d$. In some case, this component should contains interesting structure and information. For example, we have the following result:

Theorem 2.28 ([Ber+11])**.** *Let X, Y be two cubic threefolds with respective components $\mathcal{A}_X, \mathcal{A}_Y$. If there exists an equivalence $\mathcal{A}_X \simeq \mathcal{A}_Y$, then $X \simeq Y$.*

For general hypersurface $X \subset \mathbb{P}^N$, the Kuznetsov component $\mathcal{A}_X$ admits a Serre functor S, and set $c = \gcd(d, N+1)$, we have $S^{d/c} = [(N+1)(d-2)/c]$ [Kuz17]. The kind of categories is called the fractional Calabi-Yau category. The most famous case is that of a cubic fourfold, when the category $\mathcal{A}_X$ can be thought of as a non-commutative K3 surface.

Conjecture 2.29 (Kuznetsov)**.** *For a cubic fourfold X, we have $\mathcal{A}_X \simeq D(S)$ for a K3 surface S if and only if X is rational.*

3 Lecture 4

3.1 Quotient stack

3.1.1 Naive example

Let X be a set and $R \subset X \times X$ be an equivalence relation on X, this data gives us a diagram

$$R \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} X$$

where the maps are given by projections and diagonal map. This diagram defines a groupoid $[X/R]$:

- The objects of $[X/R]$ are the elements of X.
- The morphisms of $[X/R]$ are the elements of R, where the source and target are given by the projections.
- The identity map is given by the diagonal map.
- Composition of morphisms is well-defined since the relation is transitive.
- All morphisms are invertible since the relation is symmetric.

Note that the colimit of this diagram in the category of sets is precisely the set-theoretic quotient X/R, but the whole groupoid remembers how elements are identified.

Similarly, if we have a group G acting on the set X, there is a quotient groupoid X/G defined by the diagram

$$G \times X \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} X,$$

where the source map is the projection $(g, x) \mapsto x$, the target map is the action map $(g, x) \mapsto gx$, and the identity map is $x \mapsto (e, x)$. If we take the colimit of this diagram in the category of groupoids, we will get a groupoid X/G, and there is a canonical isomorphism $X \times_{X/G} X \simeq G \times X$. Informally speaking, the groupoid X/G remembers all infromation about the G-action on X.

This idea can be generalized to the other case, and the above diagram must be replaced by the simplicial group action diagram

$$\cdots \to G \times G \times X \to G \times X \to X.$$

Let $[X/G]$ be the geometric realization, i.e. the colimit in the ∞-groupoids, then this diagram is precisely the Čech nerve of the quotient map $X \to [X/G]$. In particular, there is a bijective between groupoids objects acting on X and maps $X \to Y$ which are "surjective".

From now on, in this section we will freely use the theory of ∞-categories, everything should be understand as in ∞-categories. And for simplicity, we will not draw all the arrows in the simplicial diagram.

3.1.2 Stack and group action

Definition 3.1. A stack X is a sheaf $X\colon \mathrm{Sch}^{op} \to$ Groupoid under the fpqc topology.

Definition 3.2. A morphism $f\colon U \to X$ is an effective epimorphism if the fiber product $U \times_X \cdots \times_X U$ exists, and the following simplicial diagram is a colimit diagram

$$\cdots \to U \times_X U \times_X U \to U \times_X U \to U \to X.$$

Definition 3.3. Let X be a stack and let G be a group object in Stack. A G-torsor over X is a G-equivariant morphism $\pi\colon U \to X$, where U is a stack with an action of G, and X is regarded with trivial G-action, such that

1. The map $\pi\colon U \to X$ is an effective epimorphism.
2. The canonical map $G^{\bullet}_U \to U^{\bullet}$ is invertible, where $U^{\bullet}$ is the Čech nerve of the map $\pi\colon U \to X$, and $G^{\bullet}_U$ is the action groupoid

$$\cdots \to G \times G \times X \to G \times X \to X.$$

Remark 3.4. We can identity X as the geometric realization of the following simplicial diagram

$$\cdots \to G \times G \times U \to G \times U \to U.$$

And the second condition is equivalent to say the canonical map $(\mathrm{action}, \mathrm{pr})\colon G \times U \to U \times_X U$ is invertible.

Definition 3.5. Let G be a group stack over a base scheme S, and let U be a stack over S with G-action. The quotient stack $[U/G]$ is defined as the geometric realization of the following simplicial diagram:

$$\cdots \to G \times G \times U \to G \times U \to U.$$

Remark 3.6. By construction, the quotient map $U \to [U/G]$ is a G-torsor.

We can compute the functor of points of this colimit.

Theorem 3.7. *Fix a scheme T over S, there is a canonical isomorphism between $[U/G](T)$ and the groupoid of diagrams*

$$T \xleftarrow{\pi} Y \xrightarrow{f} U,$$

where Y is an S-stack with G-action, $f\colon Y \to U$ is a G-equivariant S-morphism, and π is a G-torsor over T.

Proof. For an element $g \in [U/G](T)$, we have the following Cartesian diagram:

$$\begin{array}{ccc} Y & \xrightarrow{f} & U \\ {\scriptstyle\pi}\downarrow & \ulcorner & \downarrow \\ T & \xrightarrow{g} & [U/G] \end{array},$$

where the map π is a G-torsor since effective epimorphism is stable under the base change. The morphism f is G-equivariant since it can be lifted to a map of simplicial diagrams $G^\bullet_U \times_{[U/G]} T \to G^\bullet_U$.

Conversely, given a G-torsor $\pi\colon Y \to T$ and a G-equivariant map $f\colon Y \to U$ given by a map $f^\bullet\colon G^\bullet_Y \to G^\bullet_U$, we can recover T from the geometric realization and get the natural map

$$T \simeq [Y/G] = \operatorname{colim} G^\bullet_Y \to \operatorname{colim} G^\bullet_U = [U/G].$$

□

Example 3.8. The classifying stack of G is defined as the quotient stack

$$BG := [S/G]$$

with respect to the trivial action. By functor of points, every G-torsor $\pi\colon Y \to T$ is classified by a map $T \to BG$ fitting into following Cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & S \\ {\scriptstyle\pi}\downarrow & \ulcorner & \downarrow \\ T & \xrightarrow{g} & BG \end{array}.$$

3.1.3 Quasi-coherent sheaves on stack

Definition 3.9. We define the functor $\mathrm{Qcoh}\colon \mathrm{Stack} \to \mathrm{Cat}$ as the right Kan extension of the functor $\mathrm{Spec}\, R \mapsto \mathsf{Mod}_R$ along the inclusion $\mathsf{Ring}^{\mathrm{op}} \hookrightarrow \mathrm{Stack}$. In particular, for a stack X, we have

$$\mathrm{Qcoh}(X) := \lim_{R,x} \mathsf{Mod}_R$$

for all R and R-point $x \in X(R)$.

For an effective epimorphism $f\colon X \to Y$, we have $\mathrm{Qcoh}(Y) \simeq \lim \mathrm{Qcoh}(X^\bullet)$, so for a quotient stack $[X/G]$, we have $\mathrm{Qcoh}([X/G]) \simeq \lim \mathrm{Qcoh}(G_X^\bullet)$. Since this limit can be computed as the limit of following diagram:

$$\mathrm{Qcoh}(G \times G \times X) \leftarrow \mathrm{Qcoh}(G \times X) \leftarrow \mathrm{Qcoh}(X),$$

we can give a specific description of the objects in $\mathrm{Qcoh}([X/G])$, which we will call G-equivariant quasi-coherent sheaves on X. They are just quasi-coherent sheaves $\mathcal{F}$ on X together with a (specified) isomorphism $a^*\mathcal{F} \simeq \mathrm{pr}^*\mathcal{F}$, where $a\colon G \times X \to X$ is the action map and $\mathrm{pr}\colon G \times X \to X$ is the projection, and this isomorphism satisfying the cocycle condition on $G \times G \times X$. We can then define the derived category of a stacks as the derived category of (quasi-)coherent sheaves, or we can define it via the right Kan extension. Of curse these two constructions are equivalent.

Theorem 3.10. *Let $\mathcal{C}$ be a site with fintary topology generated by a class of morphism S and let $\mathcal{C}_0$ be a full subcategory which is closed under the pullback and finite coproducts. We say that $\mathcal{C}_0$ is a basis for $\mathcal{C}$ if for every object $X \in \mathcal{C}$ there exists some morphisms $Y_i \to X$ where $Y_i \in \mathcal{C}_0$ such that $\sqcup_i Y_i \to X \in S$.*

In this case, let F be a functor $\mathcal{C}^{op} \to \mathcal{D}$ such that D admits limits, then F is a sheaf if and only if

1. *$F_0 = F|_{\mathcal{C}_0}$ is a sheaf on $\mathcal{C}_0$,*

2. *F is the right Kan extension of F_0 along $\mathcal{C}_0 \to \mathcal{C}$.*

Any reasonable definition of derived category and quasi-coherent sheaf should make them be fpqc sheaf, so they are always given by the right Kan extension of the restriction on affine scheme.

Definition 3.11. A linear algebraic group G is called linearly reductive if the functor of G-invariants for linear representations of G,i.e.

$$-^G\colon \operatorname{Qcoh}([*/G]) \to \operatorname{Qcoh}(*), V \mapsto V^G$$

is exact.

Proposition 3.12 ([MFK94]). *Over a field of characteric 0, a linear algebraic group is linearly reductive if and only if it is reductive.*

As G is reductive, we can simplify the computation of the morphism space in the derived category of G-equivariant sheaves.

Lemma 3.13. *Assume that G is reductive and let $\mathcal{E}$ and $\mathcal{F}$ be two G-equivariant quasi-coherent sheaves on X. Then there are natural ismorphisms,*

$$\operatorname{Ext}^i_{[X/G]}(\mathcal{E}, \mathcal{F}) \simeq \operatorname{Ext}^i_X(\mathcal{E}, \mathcal{F})^G.$$

Moreover, if $\mathcal{E}, \mathcal{F}$ are complexes of G-equivariant quasi-coherent sheaves on X, then

$$\operatorname{Hom}_{[X/G]}(\mathcal{E}, \mathcal{F}[i]) \simeq \operatorname{Hom}_X(\mathcal{E}, \mathcal{F}[i])^G$$

in the category $D(k)$.

Proof. In general, we should have (in the sence of derived functor)

$$\operatorname{Hom}_{[X/G]}(-,-) = \operatorname{Hom}_X(-,-)^G.$$

We will have a Grothendieck spectral sequence to compute this functor, but in this case, the functor $-^G$ is exact, so this spectral sequence dengenerated at the second page, and we get the result. □

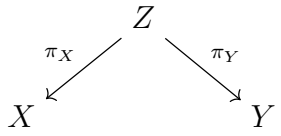
Proposition 3.14. *Let X be a smooth variety equipped with a G-action. If G is reductive, then any object of $D([X/G])$ is quasi-isomorphic to a bounded complex of G-equivariant vector bundles.*

Proof. This property is called the resolution property, Thomason proved every G-equivariant coherent sheaf on X admits a surjection from a G-equivariant vector bundle [Tho87, Lemma 2.4], which immediatedly implies the result. □

3.2 Derived equivalence and birational geometry

Many birational invariants are derived invariants (dimension, Kodaira dimension and so on), so one want to know if derived equivalent varieties are birational or birational varieties have equivalent derived categories. In general, the answer to both questions is negative (Abelian varieties, K3 surface, blow up), but there exists some precise results and conjecture concerning these ideals.

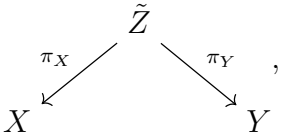
Proposition 3.15. *Let X, Y be two smooth projective varieties over an algebraically closed field. Suppose there exists an equivalence $D(X) \simeq D(Y)$. If the (anti)-canonical bundle of X is big, then X and Y are birational and , more precisely, there exists a birational correspondence*



with $\pi_X^* \omega_X \simeq \pi_Y^* \omega_Y$.

Proof. Sketch: Assume that this equivalence is given by the Fourier-Mukai transform with kernel $\mathcal{P}$, then there exists an irreducible component $Z \subset \mathrm{supp}(\mathcal{P})$ that surjects onto X, and the pullbacks (of some power of) ω_X and ω_Y on the normalization $\tilde{Z} \to Z$ coincide.

We in fact constructed a birational correspondence



and by construction we have $\pi_X^* \omega_X^r \simeq \pi_Y \omega_Y^r$ for some $r > 0$. And we have

$$\pi_X^* \omega_X + \mathcal{O}(\sum a_i E_i) \simeq \pi_Y^* \omega_Y + \mathcal{O}(\sum a_i' E_i),$$

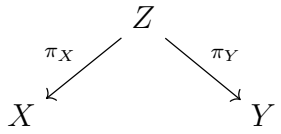
where E_i are exceptional divisors with respect to π_X and π_Y. After taking r-th power, we get

$$\mathcal{O}(r(a_i - a_i')E_i) = \mathcal{O}.$$

Thus it suffices to show a linear combination $\sum \alpha_i E_i = 0$ if and only if $\alpha_i = 0$, by some calculations we will get the result. □

Corollary 3.16. *If X and Y have equivalent derived categories and ω_X or ω_X^* is ample, then $X \simeq Y$.*

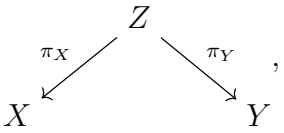
Definition 3.17. Two varieties X and Y are called K-equivalent if there exists a birational correspondence



with $\pi_X^*\omega_X \simeq \pi_Y^*\omega_Y$.

Corollary 3.18. *Two D-equivalent varieties X and Y with X of maximal Kodaira dimension are K-equivalent.*

Conjecture 3.19 (DK conjecture [Kaw02])**.** *For a K-inequality $\mu\colon X \dashrightarrow Y$, which is given by a birational map*



such that $\pi_Y K_Y \equiv \pi_X K_X + D$ for some effective divisor D, there is a fullt faithful functor $\Phi\colon D(X) \hookrightarrow D(Y)$. Moreover, assume that μ is a K-equivalence,i.e. $D = 0$, then Φ is an equivalence.

We will now introduce two standard examples, and the detailed calculation can be found in [Huy06].

3.2.1 Standard flip

Suppose X contains a smooth variety Y isomorphic to $\mathbb{P}^k$ and such that the normal bundle $\mathcal{N} := \mathcal{N}_{\mathbb{P}^k/X}$ is isomorphic to $\mathcal{O}(-1)^{l+1}$. By the adjunction formula for $\mathbb{P}^k \subset X$ we get $\omega_X \mid_{\mathbb{P}^k} = \mathcal{O}(l-k)$.

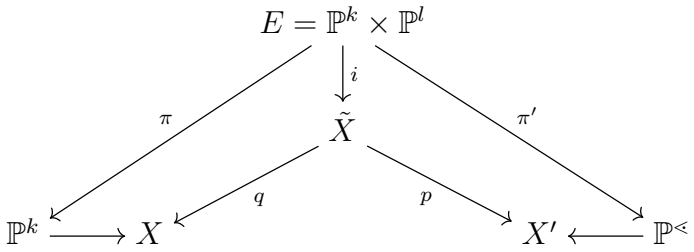
The blow up $q\colon \tilde{X} \to X$ of X along $\mathbb{P}^k$ produces an exceptional divisor $E \simeq \mathbb{P}(\mathcal{N}) \simeq \mathbb{P}^k \times \mathbb{P}^l$. The two projections will be denoted by $\pi\colon \mathbb{P}^k \times \mathbb{P}^l \to \mathbb{P}^k$ and $\pi'\colon \mathbb{P}^k \times \mathbb{P}^l \to \mathbb{P}^l$.

By Fujiki-Nakano criterion, there exists a contraction $p\colon \tilde{X} \to X'$, which is a blow up of $\mathbb{P}^l \subset X'$ with exceptional divisor E such that the restriction of p to E equals π', and we will assume that X' is projective.

When $l < k$, the situation is not symmetric, for $l < k$ we have the restriction of ω_X on $\mathbb{P}^k$ is the negtaive line bundle $\mathcal{O}(l-k)$, and the restriction of $\omega_{X'}$ on $\mathbb{P}^l$ is the positive line bundle $\mathcal{O}(k-l)$. And we have

$$q^*\omega_X \otimes \mathcal{O}(lE) \simeq \omega_{\tilde{X}} \simeq p^*\omega_{X'} \otimes \mathcal{O}(kE).$$

This diagram we just constructed



is called a standard flip. In the special case $k = l$, this is called the standard flop.

In general, flips are birational transformations which are performed in order to increase the positivity of the canonical bundle (and eventually to reach a minimal model). general flops are birational transformation that describe the passage from one minimal model to another one. In particular, under a flop the positivity of the canonical bundle will not cahnge.

Remark 3.20 (Fujiki-Nakano criterion). If $\pi\colon E = \mathbb{P}(\mathcal{N}) \to Y$ is a projective bundle over a smooth variety Y and $E \subset \tilde{X}$ is a closed embedding of codimension 1 such that $\mathcal{O}(E)$ on the fibres of $E \to Y$ is of degree -1, then the projection π extends to a morphism $q\colon \tilde{X} \to X$ onto a smooth variety X containing Y such that q is the blow up of Y.

This diagram allows us to compare the derived categories of X and X' inside the bigger derived category $D(\tilde{X})$.

Proposition 3.21 (Bondal,Orlov)**.** *Let $X \leftarrow \tilde{X} \to X'$ be the standard flip with $l \leq k$. Then*

$$q_* \circ p^*\colon D(X') \to D(X)$$

is fully faithful. If $k = l$, it defines an equivalence.

Remark 3.22. Note that this functor if just the Fourier-Mukai transform with kernel $\mathcal{O}_{\tilde{X}}$.

3.2.2 Mukai flop

The Mukai flop is quite similar to the standard flop, in this case, the derived category still does not change but the the equivalence is no longer induced by the birational correspondence. Suppose X is a smooth proejctive variety of dimension $2n$ containing a smooth subvariety $P \subset X$ if self isomorphic to $\mathbb{P}^n$, and we assume that $n \geq 1$, the normal bundle $\mathcal{N} := \mathcal{N}_{P/X}$ is isomorphic to the cotangent bundle Ω_p.

Blowing-up $P \subset X$ yields a morphism $q\colon \tilde{X} \to X$ with exceptional divisor

$$\pi\colon E \simeq \mathbb{P}(\mathcal{N}) \simeq \mathbb{P}(\Omega_P) \to P.$$

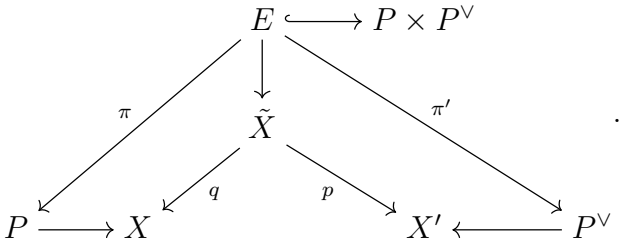
By Euler sequence, we have (assume that $P = \mathbb{P}(V)$ for some vector space V):

$$0 \to \Omega_P \to V^\vee \otimes \mathcal{O}(-1) \to \mathcal{O} \to 0.$$

Hence we get

$$\mathbb{P}(\Omega_P) \subset \mathbb{P}(V^\vee \otimes \mathcal{O}(-1)) = \mathbb{P}(V) \times \mathbb{P}(V^\vee) = P \times P^\vee.$$

By Fujiki-Nakano criterion, there exists a blow down map $p\colon \tilde{X} \to X'$, whose restriction to E is the second projection $E \subset P \times P^\vee \to P^\vee$. We have following diagram:



Remark 3.23. The most interesting example is given by the embedding of a proejctive space $\mathbb{P}^n$ into an algebraic symplectic variety X of dimension $2n$, in this case, our assumption is automatically satisfied.

Of course, this example should be consistent with the DK conjecture, i.e. there is an equivalence $D(X) \simeq D(X')$. Everything looks like the standard flop, one might except there should be a straightforward generalization. But the following negative result shows the obvious kernel does not work.

Proposition 3.24 (Kawamata,Namikawa)**.** *The Fourier-Mukai transform given by*

$$p_* \circ q^* \colon D(X) \to D(X')$$

is not fully faithful.

Proposition 3.25. *Let $Z \subset X \times X'$ be the reduced subvariety $Z := \tilde{X} \cup (P \times P^\vee)$, then*

$$\Phi_{\mathcal{O}_Z} \colon D(X) \to D(X')$$

is an equivalence.

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