Fourier–Mukai transforms and derived equivalence

Study notes

Shuai Zong

Scope

These notes develop the author's accounts of Fourier–Mukai transforms for curves, cohomological invariants of K3 surfaces, and stability conditions on cubic fourfolds. The results quoted from the literature are distinguished from the calculations reproduced here. Functors between derived categories over a field are understood to be linear over that field, and pullbacks, pushforwards and tensor products in derived categories are derived.

The K3 moduli and crystalline Torelli arguments, the cubic categorical calculations, and the deformation and integral-transform appendices are included in mathematical dependency order. Deep inputs are cited where they enter a proof. The remaining proposed extension beyond perfect stacks is explicitly separated from the established statements. The project README and mathematical review record give the source map and correction history.

Contents

Scope i

1 Derived equivalence for curves 1
- 1.1 Fourier-Mukai Transforms . . . 1
- 1.2 Derived categories of curves . . . 2
  - 1.2.1 Curve with (anti)-ample canonical bundle . . . 2
  - 1.2.2 Elliptic curve over characteristic 0 . . . 3

2 K3 invariants 8
- 2.1 Cohomological invariants of K3 surfaces . . . 8
- 2.2 Mukai-Hodge structures . . . 9
- 2.3 Derived equivalence of K3 surfaces . . . 11

3 K3 moduli and Torelli 13
- 3.1 Stable sheaves and isotropic Mukai vectors . . . 13
- 3.2 Normalizing and lifting an equivalence . . . 15
- 3.3 Partner counts . . . 17
- 3.4 Crystalline Torelli and characteristic subspaces . . . 17

4 Fractional Calabi–Yau categories 20
- 4.1 Adjunctions and Lefschetz decompositions . . . 20
- 4.2 Hypersurfaces . . . 21

5 Cubic stability 23
- 5.1 Bridgeland stability . . . 24

6 Cubic moduli and Torelli 27
- 6.1 The Mukai lattice and stability space . . . 27

6.2 The line-independence argument 28
6.3 Small Ext groups and stable objects 29
6.4 Lines as a moduli space 30
6.5 Categorical Torelli and classical Torelli 30

A F-crystals and height **32**

B Deformations of K3 surfaces **35**
B.1 Preserving a line bundle 35
B.2 Projective lifting 36

C Etale cohomology calculations **37**
C.1 Kummer sequence and curves 37
C.2 Purity, affine space and projective space 38
C.3 Artin–Schreier in characteristic p 38

D Integral transforms and linear categories **40**
D.1 The affine calculation 40
D.2 Perfect stacks and external products 41
D.3 The bar and monad comparison 42
D.4 Beyond perfect stacks 43

Chapter 1

Derived equivalence for curves

This chapter is based on [Huy06].

1.1 Fourier-Mukai Transforms

Let X and Y be smooth projective varieties and denote the two projections by

$$q\colon X \times Y \to X, p\colon X \times Y \to Y.$$

Definition 1.1.1. *Let $\mathcal{P} \in D^b(X \times Y)$, we call it as the Fourier-Mukai kernel, and its associated Fourier-Mukai transform is the functor*

$$\begin{aligned} \Phi_{\mathcal{P}}\colon D^b(X) &\to D^b(Y) \\ \mathcal{E}^\bullet &\mapsto p_*(q^*\mathcal{E}^\bullet \otimes \mathcal{P}). \end{aligned}$$

The Fourier-Mukai transforms have many fine properties. The composition of Fourier-Mukai transforms is still Fourier-Mukai transform, the Fourier-Mukai transform has both left adjoint and right adjoint which are Fourier-Mukai transforms, too. And both these transforms have kernel which can be described explicitly. The following celebrated theorem tells us that the Fourier-Mukai transforms are the essential functor we should care about in geometric case.

Theorem 1.1.2. *Let X and Y be smooth projective varieties and assume we have a fully faithful exact functor*

$$F\colon D^b(X) \to D^b(Y).$$

Then there exists a unique object $\mathcal{P} \in D^b(X \times Y)$ up to isomorphism such that $F \simeq \Phi_{\mathcal{P}}$.

This theorem is quoted here. The separate draft on integral transforms does not supply a completed proof of this triangulated representability statement.

Remark 1.1.3. *The first proof of this theorem is given in the [Orl03], though it assumed that F has a right adjoint, it follows from [BB02] that every triangulated fully faithful functor admits a right adjoint.*

1.2 Derived categories of curves

1.2.1 Curve with (anti)-ample canonical bundle

Proposition 1.2.1. *Let X and Y be smooth projective varieties over a field k. If there exists an exact equivalence of their bounded derived categories. Then $\dim X = \dim Y$.*

Proof. We know that any equivalence $F\colon D^b(X) \to D^b(Y)$ is of the form $\Phi_{\mathcal{P}}$ for some $\mathcal{P} \in D^b(X \times Y)$. Moreover, F has left adjoint with kernel $\mathcal{P}_L = \mathcal{P}^\vee \otimes p^*\omega_Y[\dim Y]$ and right adjoint with kernel $\mathcal{P}_R = \mathcal{P}^\vee \otimes q^*\omega_X[\dim X]$.

Since F is an equivalence, its right and left adjoint are both quasi-inverse to F. By the uniqueness of the Fourier-Mukai kernel, we conclude that $\mathcal{P}_L \simeq \mathcal{P}_R$. Hence,

$$\mathcal{P}^\vee \simeq \mathcal{P}^\vee \otimes (q^*\omega_X \otimes p^*\omega_Y^{-1})[\dim X - \dim Y].$$

The nonzero bounded complex $\mathcal{P}^\vee$ cannot be isomorphic to a nonzero shift of itself tensored with a line bundle: tensoring preserves its lowest and highest nonzero cohomology degrees. Hence $\dim X = \dim Y$. □

Proposition 1.2.2. *Let X and Y be smooth projective varieties over a field k. If there exists an exact equivalence of their bounded derived categories. Then $R(X) \simeq R(Y)$, where $R(X), R(Y)$ means the canonical ring.*

Proof. Let $\mathcal{P}$ be the kernel of the equivalence and let $\mathcal{Q} \in D^b(Y \times X)$ be the kernel of its inverse. Write $\star$ for convolution, in the order of composition of functors. Convolution defines an equivalence

$$D^b(X \times X) \longrightarrow D^b(Y \times Y), \qquad \mathcal{R} \longmapsto \mathcal{P} \star \mathcal{R} \star \mathcal{Q}.$$

Its inverse is convolution with $\mathcal{Q}$ on the left and $\mathcal{P}$ on the right, since the two inverse kernels convolve to the diagonal kernels. This equivalence preserves convolution and its unit.

Let ι_X and ι_Y be the diagonal embeddings and put $n = \dim X = \dim Y$. The kernel $\iota_{X*}\omega_X^l$ represents $S_X^l[-ln]$. An equivalence intertwines Serre functors, so uniqueness of the kernel of an equivalence identifies its image with $\iota_{Y*}\omega_Y^l$, compatibly with tensor powers. We therefore obtain

$$H^0(X, \omega_X^l) = \mathrm{Hom}_{X\times X}(\mathcal{O}_{\Delta_X}, \iota_{X*}\omega_X^l) \simeq \mathrm{Hom}_{Y\times Y}(\mathcal{O}_{\Delta_Y}, \iota_{Y*}\omega_Y^l) = H^0(Y, \omega_Y^l).$$

Multiplication of sections corresponds to convolution of the displayed morphisms. The equivalence consequently gives an isomorphism of graded rings $R(X) \simeq R(Y)$. □

Proposition 1.2.3 (Bondal,Orlov)**.** *Let X and Y be smooth projective varieties and assume that the canonical bundle of X is (anti)-ample. If we have an exact equivalence $D^b(X) \simeq D^b(Y)$, then X and Y are isomorphic.*

Proof. One can show that the canonical bundle of Y is also (anti)-ample ([4.11 Huy06, p. 95]). Since X and Y are projective, we get $X = \mathrm{Proj}\, R(X) \simeq \mathrm{Proj}\, R(Y) = Y$ when they have ample canonical bundle. Similarly, we have $X = \mathrm{Proj}\, R(X, \omega_X^*) \simeq \mathrm{Proj}\, R(Y, \omega_Y^*) = Y$ when they have anti-ample canonical bundle. □

Corollary 1.2.4. *Let C be a smooth projective curve over a field k with genus $g \neq 1$, and let Y be a smooth projective variety with an exact equivalence $D^b(C) \simeq D^b(Y)$, then Y is isomorphic to C.*

Proof. By Riemann-Roch theorem, if C has genus $g = 0$, then its canonical bundle has degree -2 and is anti-ample, if C has genus $g > 1$, then it has ample canonical bundle. □

1.2.2 Elliptic curve over characteristic 0

When we consider the elliptic curves, we can not directly reconstruct it from derived category. But the Fourier-Mukai transforms between the derived category can induce the morphism between K_0 group, and we can pass from K_0 down to cohomology via Chern character. The analogous cohomology Fourier-Mukai transform preserves the Hodge structure, since the elliptic curve is determined by its weight-1 Hodge structure, we can get all information.

For a smooth projective variety X, we have the natural map

$$\begin{aligned} D^b(X) &\to K_0(X) \\ \mathcal{F}^\bullet &\mapsto [\mathcal{F}^\bullet] = \sum (-1)^i [\mathcal{F}^i] \end{aligned}$$

As any coherent sheaf over X has finite locally free resolution, the elements of $K_0(X)$ can be written as linear combinations of locally free sheaves $\mathcal{E}_i$. This gives a ring structure on $K_0(X)$ by setting $[\mathcal{E}_1] \cdot [\mathcal{E}_2] := [\mathcal{E}_1 \otimes \mathcal{E}_2]$. Since the derived tensor product of complexes is the ordinary tensor product for complex of locally free sheaves, the map $D^b(X) \to K_0(X)$ is compatible with the multiplicative structure.

We can introduce K-theoretic Fourier-Mukai transform as an analogy. Let $e \in K_0(X \times Y)$, where X and Y are smooth projective varieties, we define

$$\begin{aligned} \Phi_e^K \colon K_0(X) &\to K_0(Y) \\ f &\mapsto p_!(e \otimes q^*(f)). \end{aligned}$$

The two Fourier-Mukai transform commute:

$$\begin{array}{ccc} D^b(X) & \xrightarrow{\Phi_{\mathcal{P}}} & D^b(Y) \\ \downarrow & & \downarrow \\ K_0(X) & \xrightarrow[\Phi_{[\mathcal{P}]}]{} & K_0(Y) \end{array}$$

From now on, we assume all varieties have ground field $\mathbb{C}$. We can use Grothendieck-Riemann-Roch to pass from $K_0(X)$ down to (rational) cohomology.

Theorem 1.2.5. *Let $f \colon X \to Y$ be a morphism of smooth projective varieties over a field. Then for $e \in K_0(X)$ we have*

$$f_*(\mathrm{ch}(e) \cdot \mathrm{td}(X)) = \mathrm{ch}(f_!(e)) \cdot \mathrm{td}(Y).$$

The Hirzebruch-Riemann-Roch is a special case of the structure morphism $f \colon X \to \mathrm{Spec}(k)$:

Corollary 1.2.6. *For any $e \in K_0(X)$ we have*

$$\chi(e) = \int_X \mathrm{ch}(e) \cdot \mathrm{td}(X).$$

Definition 1.2.7. *The Mukai vector of a class $e \in K_0(X)$ is defined as the cohomology class*

$$v(e) = \mathrm{ch}(e) \cdot \sqrt{\mathrm{td}(X)}.$$

Clearly, by definition the induced map $v\colon K_0(X) \to H^*(X, \mathbb{Q})$ is additive.

For any cohomology class $\alpha \in H^*(X \times Y, \mathbb{Q})$ we introduces the cohomological Fourier-Mukai transforms:

$$\begin{aligned}\Phi_\alpha^H\colon H^*(X, \mathbb{Q}) &\to H^*(Y, \mathbb{Q}) \\ \beta &\mapsto p_*(\alpha \cdot q^*(\beta)).\end{aligned}$$

By the Grothendieck-Riemann-Roch theorem, we can easily deduce that

Proposition 1.2.8. *Let $e \in K_0(X \times Y)$. then the following diagram commutes:*

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\Phi_e^K} & K_0(Y) \\ \downarrow & & \downarrow \\ H^*(X, \mathbb{Q}) & \xrightarrow[\Phi_{v(e)}^H]{} & H^*(Y, \mathbb{Q}) \end{array}$$

Let $\mathcal{P} \in D^b(X \times Y)$ be the kernel of a Fourier-Mukai transform $\Phi_\mathcal{P}$, we denote the induced cohomological Fourier-Mukai transform $\Phi_{v(\mathcal{P})}^H$ by $\Phi_\mathcal{P}^H$.

Proposition 1.2.9. *If $\mathcal{P} \in D^b(X \times Y)$ gives an equivalence $\Phi_\mathcal{P}\colon D^b(X) \to D^b(Y)$, then the induced cohomological Fourier-Mukai transform $\Phi_\mathcal{P}^H$ is a bijection of rational vector space.*

Proof. If $\Phi_\mathcal{P}$ is an equivalence, then

$$\Phi_{\mathcal{P}_R} \circ \Phi_\mathcal{P} \simeq \Phi_{\mathcal{O}_\Delta}, \Phi_\mathcal{P} \circ \Phi_{\mathcal{P}_R} \simeq \Phi_{\mathcal{O}_\Delta}$$

Thus, we conclude that

$$\Phi_{\mathcal{P}_R}^H \circ \Phi_\mathcal{P}^H = \Phi_{\mathcal{O}_\Delta}^H, \Phi_\mathcal{P}^H \circ \Phi_{\mathcal{P}_R}^H = \Phi_{\mathcal{O}_\Delta}^H$$

So it suffices to show that $\Phi_{\mathcal{O}_\Delta}^H = id$. Use the Grothendieck-Riemann-Roch formula for the diagonal embedding $\iota\colon X \to X \times X$:

$$\mathrm{ch}(\mathcal{O}_\Delta) \cdot \mathrm{td}(X \times X) = \iota_*(\mathrm{ch}(\mathcal{O}_X) \cdot \mathrm{td}(X)) = \iota_* \mathrm{td}(X).$$

Notice that $\iota^* \mathrm{td}(X \times X) = \mathrm{td}(\iota^* T_{X \times X}) = \mathrm{td}(T_X \oplus T_X) = \mathrm{td}(X) \cdot \mathrm{td}(X)$. Then dividing by $\sqrt{\mathrm{td}(X \times X)}$ yields

$$\mathrm{ch}(\mathcal{O}_\Delta) \cdot \sqrt{\mathrm{td}(X \times X)} = \iota_*(1).$$

Hence we get

$$\Phi_{\mathcal{O}_\Delta}^H(\beta) = p_*(q^*(\beta) \cdot \iota_*(1)) = p_*(\iota_*(\iota^* q^*(\beta))) = \beta.$$

□

Now a natural question is, does this induced equivalence preserve the Hodge structure? Since the Chern classes are classes of type (p,p), the Mukai vector factorizes over the algebraic part of the cohomology

$$v() = \mathrm{ch}()\sqrt{\mathrm{td}(X)}\colon K_0(X) \to \bigoplus H^{p,p}(X) \cap H^{2p}(X,\mathbb{Q}).$$

We can get

Proposition 1.2.10. *If $\mathcal{P} \in D^b(X \times Y)$ gives an equivalence $\Phi_{\mathcal{P}}\colon D^b(X) \to D^b(Y)$, then the induced cohomological Fourier-Mukai transform $\Phi^H_{\mathcal{P}}$ yields isomorphisms*

$$\bigoplus_{p-q=i} H^{p,q}(X) \simeq \bigoplus_{p-q=i} H^{p,q}(Y)$$

for $i = -\dim X, \cdots, 0, \cdots, \dim X$.

Proof. It suffices to show that

$$\Phi^H_{\mathcal{P}}(H^{p,q}(X)) \subset \bigoplus_{r-s=p-q} H^{r,s}(Y).$$

By Kunneth theorem we have $H^*(X \times Y, \mathbb{Q}) \simeq H^*(X,\mathbb{Q}) \otimes H^*(Y,\mathbb{Q})$, so we can write $\mathrm{ch}(\mathcal{P}) \cdot \sqrt{\mathrm{td}(X \times Y)}$ as $\sum \alpha^{p',q'} \otimes \beta^{r,s}$ with $\alpha^{p',q'} \in H^{p',q'}(X)$ and $\beta^{r,s} \in H^{r,s}(Y)$. Moreover, we have $p' + r = q' + s$. For $\alpha \in H^{p,q}(X)$, we have

$$\Phi^H_{\mathcal{P}}(\alpha) = \sum (\int_X \alpha \wedge \alpha^{p',q'})\beta^{r,s} \in \bigoplus H^{r,s}(Y).$$

Hence, $p - q = q' - p' = r - s$. □

Proposition 1.2.11. *Let E and E' be two complex elliptic curves with equivalent bounded derived categories, then they are isomorphic.*

Proof. Suppose $\Phi_{\mathcal{P}}\colon D^b(E) \to D^b(E')$ is an equivalence, then the induced cohomological Fourier-Mukai transform $\Phi^H_{\mathcal{P}}$ give us

$$H^1(E,\mathbb{Q}) \simeq H^1(E',\mathbb{Q}), (H^0 \oplus H^2)(E,\mathbb{Q}) \simeq (H^0 \oplus H^2)(E',\mathbb{Q}).$$

We know that $\Phi^H_{\mathcal{P}}\colon H^1(E,\mathbb{Q}) \to H^1(E',\mathbb{Q})$ respects the Hodge decomposition $H^1 = H^{1,0} \oplus H^{0,1}$. Since the elliptic curve is determined by its weight-1 Hodge structure, it suffices to show that $\Phi^H_{\mathcal{P}}$ is defined over $\mathbb{Z}$.

We have $\mathrm{ch}(\mathcal{P}) = r + c_1(\mathcal{P}) + (c_1^2 - 2c_2)(\mathcal{P})/2$ and $\mathrm{td}(E \times E') = p^*(1) \cdot q^*(1) = 1$, and the degree 4 term in $\mathrm{ch}(\mathcal{P})$ does not contribute to $H^1(E) \to H^1(E')$. Thus the map on H^1 is induced by the integral class $c_1(\mathcal{P})$. The inverse Fourier–Mukai transform has the same integrality property, so this is an isomorphism of integral Hodge structures. It determines an isomorphism of the associated one-dimensional complex tori, hence of the elliptic curves. □

Corollary 1.2.12. *Let C be a smooth complex projective curve, and let Y be a smooth complex projective variety with an exact equivalence $D^b(C) \simeq D^b(Y)$, then Y is isomorphic to C.*

Remark 1.2.13. *For higher-genus curves, Torelli uses the principal polarization on the Jacobian. An isomorphism of integral Hodge structures alone does not supply that polarization compatibility. The preceding corollary instead uses reconstruction from the ample canonical bundle when the genus is greater than one.*

Remark 1.2.14. *In general, let C and C' be two smooth projective genus 1 curves over a field k with equivalent derived categories, we can only get they have the same Jacobian [AKW14].*

The positive-characteristic genus-one case is not treated here.

Chapter 2

Cohomological invariants of K3 surfaces

Throughout this chapter, surfaces are geometrically connected and smooth projective. For K3 surfaces the ground field is algebraically closed; in positive characteristic we assume $p > 2$. The crystalline statements use $W = W(k)$ and $K = W[1/p]$. The cohomological background is quoted from [Huy16]; the computations of induced transforms are included below. The classification of smooth projective minimal surfaces with numerically trivial canonical bundle gives abelian, K3, Enriques and (quasi-)hyperelliptic surfaces. Outside characteristics two and three, triviality of the canonical bundle leaves the abelian and K3 cases. The additional condition $H^1(\mathcal{O}_X) = 0$ distinguishes the K3 case considered here; in the small characteristics the classification requires its exceptional cases [Huy16].

2.1 Cohomological invariants of K3 surfaces

Definition 2.1.1. *A K3 surface is a smooth and projective surface X over a field k such that $\omega_X \simeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.*

Now we will compute cohomological invariants for a K3 surface X:

l-adic betti number Since X is a surface, we have $b_0 = b_4 = 1$. Since $H^1(\mathcal{O}_X)$ is the Zariski tangent space of Pic^0_X at the identity, the triviality of $H^1(\mathcal{O}_X)$ shows Pic^0_X is trivial. Thus also Albanese variety, which is the dual of Picard, is trivial. So $b_1 = 2 \dim \mathrm{Alb}(X) = 0$, by Poincare duality we get

$b_3 = b_1 = 0$. Then by the Noether formula we get $12\chi(\mathcal{O}_X) = 24 = c_1^2 + c_2 = c_2$, implies $b_2(X) = 22$.

Hodge number $h^{0,0} = h^{2,2} = 1$, $h^{2,0} = h^{0,2} = 1$, $h^{0,1} = 0$ by the definition. Then Serre duality gives $h_{0,1} = h^{2,1} = 0$ and $h^{1,0} = h^{1,2}$. Since over X we have the non-canonical isomorphism $T_X \simeq \Omega^1_X$, we get $H^{1,0}(X) = H^0(X, \Omega^1_X) = H^0(X, T_X) = 0$. The Hirzebruch–Riemann–Roch theorem shows that $h^{1,1} = c_2(\Omega_X) - 4 = 20$.

Remark 2.1.2. *$H^0(X, T_X) = 0$ i.e. a K3 surface has no global vector fields. This theorem is trivial when the base field is $\mathbb{C}$, but in positive characteristic there is no easy proof.*

Proposition 2.1.3. *Let X be a K3 surface over an algebraically closed field of characteristic $p > 2$. The Hodge-to-de Rham spectral sequence*

$$E_1^{i,j} = H^j(X, \Omega^i_X) \Rightarrow H^{i+j}_{dR}(X/k)$$

degenerates, and $H^n_{crys}(X/W)$ is free of rank b_n.

Proof. The vanishing $H^2(X, T_X) = 0$ makes the deformation problem unobstructed, so X lifts to $W_2(k)$. Deligne–Illusie applies because $\dim X = 2 < p$, and gives degeneration. To check integral freeness, write a_n for the number of cyclic torsion summands of $H^n_{crys}(X/W)$. Derived crystalline base change and the universal coefficient sequence give

$$\dim_k H^n_{dR}(X/k) = b_n + a_n + a_{n+1}.$$

Here the rational crystalline ranks are the Betti numbers. The Hodge numbers computed above and degeneration give $\dim H^n_{dR} = b_n$ for every n. Thus every a_n vanishes. The comparison and degeneration theorems are the quoted inputs; this completes the original lifting argument [Huy16]. □

2.2 Mukai-Hodge structures

Let H be a Weil cohomology with coefficients in a characteristic 0 field k. We will denote the i-th cohomology group of the scheme X twisted by n as $H^i(X/k)(n)$.

For a dimension d smooth, projective variety X, its even and odd Mukai-Hodge structures are defined by:

$$H^{even}(X/k) = \bigoplus_{i=0}^{d} H^{2i}(X/k)(i)$$
$$H^{odd}(X/k) = \bigoplus_{i=0}^{d} H^{2i-1}(X/k)(i).$$

For two varieties X and Y with dimension d_X and d_Y respectly, a Fourier-Mukai transforms $\Phi_{\mathcal{P}}\colon D^b(X) \to D^b(Y)$ with kernel $\mathcal{P} \in D^b(X \times Y)$ induces the Fourier-Mukai transforms $\Phi_{\mathcal{P}}^{CH} = p_{2*}(v(\mathcal{P}) \cup p_1^*(-))$ on Chow groups, where $v(\mathcal{P})$ is the Mukai vector of $\mathcal{P}$. By the cycle class map $\gamma\colon CH^i(X) \to H^{2i}(X)(i)$ of the Weil cohomology H, we have the induced map on the level of cohomology. To be specific, this map is defined as the composition:

$$\Phi_{\mathcal{P}}^{i,j} = p_{2*}(\gamma(v^j(\mathcal{P})) \cup p_1^*(-))\colon H^i(X/k) \to H^{i+2(j-d_X)}(Y/k)(j-d_X).$$

We have the totally map on the Mukai-Hodge structure:

$$\Phi_{\mathcal{P}}^{even} := \bigoplus_{i=0}^{d_X} \sum_{j=0}^{d_X+d_Y} \Phi_{\mathcal{P}}^{2i,j}(i)\colon H^{even}(X/k) \to H^{even}(Y/k),$$
$$\Phi_{\mathcal{P}}^{odd} := \bigoplus_{i=0}^{d_X} \sum_{j=0}^{d_X+d_Y} \Phi_{\mathcal{P}}^{2i-1,j}(i)\colon H^{odd}(X/k) \to H^{odd}(Y/k).$$

Proposition 2.2.1. *If $\mathcal{P} \in D^b(X \times Y)$ gives an equivalence $\Phi_{\mathcal{P}}\colon D^b(X) \to D^b(Y)$, then the induced maps $\Phi_{\mathcal{P}}^{even}$ and $\Phi_{\mathcal{P}}^{odd}$ are isomorphisms.*

Proof. It is as same as the proof of 1.2.9, we just need to show $\Phi_{\mathcal{O}_\Delta}^{even}$ and $\Phi_{\mathcal{O}_\Delta}^{odd}$ act identically, which is a direct consequence of the Grothendieck-Riemann-Roch. □

Remark 2.2.2. *The choice of weight for the even and odd Mukai–Hodge structures was an arbitrary one. For convenience, for a surface X we will assume its weight of even Mukai-Hodge structure is*

$$H^{even}(X/k) = H^0(X)(-1) \oplus H^2(X) \oplus H^4(X)(1).$$

Corollary 2.2.3. *Let X and Y be smooth geometrically connected projective surfaces over a finite field $\mathbb{F}_q$, where $q = p^n$ for some prime p. If $\mathcal{P} \in D^b(X \times Y)$ gives an equivalence $\Phi_{\mathcal{P}} \colon D^b(X) \to D^b(Y)$, then X and Y have the same zeta function.*

Proof. It suffices to prove every power of the Frobenius map ϕ has the same trace on $H^i(X/k)$ and $H^i(Y/k)$ for some Weil cohomology theory H. By the proof the Weil conjecture, we can work on the l-adic cohomology $H^n_{et}(X/\mathbb{Q}_l)$ for some prime $l \neq p$.

Since $\mathcal{P}$ is defined over $\mathbb{F}_q$, the isomorphism between Mukai-Hodge structures commutes with Frobenius action. Consider $H^{even}(X) = H^{even}(Y)$ and $H^{odd}(X) = H^{odd}(Y)$, the trace of the Frobenius action on both sides is the same. Note the traces of Frobenius action on H^0 and H^4 are trivially equal, and by Hard Lefschetz we have $\mathrm{Tr}(\phi, H^3) = q\,\mathrm{Tr}(\phi, H^1)$, the same argument for every ϕ^m, with q replaced by q^m, gives equality of zeta functions.

□

2.3 Derived equivalence of K3 surfaces

Proposition 2.3.1. *Let X be a K3 surface, and let Y be a smooth projective variety. If $\mathcal{P} \in D^b(X \times Y)$ gives an equivalence $\Phi_{\mathcal{P}} \colon D^b(X) \to D^b(Y)$, then Y is a K3 surface.*

Proof. Since equivalence commutes with the Serre functor, we can see the Serre functor in $D^b(Y)$ is given by shift twice, so Y has dimension 2 and trivial canonical bundle.

Then it suffices to show $H^1(Y, \mathcal{O}_Y) = 0$, consider the spectral sequence

$$E_2^{i,j} = H^i(X, \mathcal{E}xt^j_q(\mathcal{P}, \mathcal{P})) \Rightarrow \mathrm{Ext}^{i+j}_{X\times Y}(\mathcal{P}, \mathcal{P})$$
$$E_2^{i,j} = H^i(Y, \mathcal{E}xt^j_p(\mathcal{P}, \mathcal{P})) \Rightarrow \mathrm{Ext}^{i+j}_{X\times Y}(\mathcal{P}, \mathcal{P})$$

where p and q are two projections and $\mathcal{E}xt^j_p(\mathcal{P}, \mathcal{P}) = R^j p_*(\mathcal{P}^\vee \otimes \mathcal{P})$ is the relative Ext-sheaf. We can deduce the following exact sequence

$$0 \to H^1(X, \mathcal{E}xt^0_q(\mathcal{P}, \mathcal{P})) \to \mathrm{Ext}^1_{X\times Y}(\mathcal{P}, \mathcal{P}) \to H^0(X, \mathcal{E}xt^1_q(\mathcal{P}, \mathcal{P}))$$

Form [Huy06, Section 11.1] we can see $\mathcal{E}xt^0_q(\mathcal{P}, \mathcal{P}) \simeq \mathcal{O}_X$ and $\mathcal{E}xt^1_q(\mathcal{P}, \mathcal{P}) \simeq T_X$, so we get $\mathrm{Ext}^1_{X\times Y}(\mathcal{P}, \mathcal{P}) = 0$. By the analogous exact sequence we find $H^1(Y, \mathcal{O}_Y) = 0$. □

Proposition 2.3.2. *Let X and Y be K3 surfaces over an algebraically closed field of characteristic $p > 2$. A Fourier–Mukai equivalence preserves their heights, including infinite height.*

Proof. The equivalence induces an isomorphism of Mukai F-isocrystals. The summands $H^0_{crys}(X/K)(-1)$ and $H^4_{crys}(X/K)(1)$ both have slope one, so

$$H^{even}_{crys}(X/K)_{[0,1)} = H^2_{crys}(X/K)_{[0,1)}.$$

The same holds for Y, and the two slope-below-one spaces have equal dimension. For a finite-height K3 surface that dimension is its height. If the dimension is zero, the surface is supersingular and its height is ∞. Thus both the finite height and the supersingular case are preserved. The slope convention is recalled in Appendix A. □

The next chapter develops moduli spaces and partner counts; the integral supersingular Mukai crystal used there requires more than the isocrystal calculation used to preserve height.

Chapter 3

Moduli spaces and Torelli for K3 surfaces

Throughout this chapter X is a projective K3 surface over an algebraically closed field of characteristic zero or $p > 2$. The positive-characteristic lifting inputs are those of [LO14]; Appendix B explains the polarized deformation calculation.

3.1 Stable sheaves and isotropic Mukai vectors

For an ample divisor H, the projective Gieseker moduli space $M_H(v)$ parametrizes S-equivalence classes of semistable sheaves of Mukai vector v; its open stable locus is $M_H(v)^s$. This is a coarse moduli statement, not representability of the functor of families up to ordinary isomorphism. Families differing by a line bundle from the base define the same moduli morphism. Existence of the coarse space is quoted from [HL10].

The numerical Grothendieck group has the description

$$N(X) = \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}, \qquad v(E) = (r, c_1, r + c_1^2/2 - c_2).$$

The Mukai pairing and the Hilbert polynomial are

$$\begin{aligned} \langle (r,l,s), (r',l',s') \rangle &= l \cdot l' - rs' - r's, \\ \chi(E,F) &= -\langle v(E), v(F) \rangle, \\ P(E,m) &= -\langle v(E), v(\mathcal{O}_X(-mH)) \rangle. \end{aligned}$$

For a stable sheaf, $\mathrm{Hom}(E,E) = k$ and Serre duality gives $\mathrm{Ext}^2(E,E) = k$, hence

$$\dim \mathrm{Ext}^1(E,E) = v(E)^2 + 2.$$

The trace-free obstruction space vanishes: the trace on Ext^2 is dual to $H^0(\mathcal{O}_X) \to \mathrm{Hom}(E,E)$, which sends 1 to the identity. Thus the stable moduli space is smooth of this dimension. The trace of the Yoneda pairing is alternating and nondegenerate; its global form is the Mukai symplectic form.

A vector $v = (r,l,s)$ is positive if $r > 0$, or if $r = 0$ and l is nonzero effective, or if $r = l = 0$ and $s > 0$. For a primitive positive vector on a complex K3 surface, with $v^2 \geq -2$ and a v-generic polarization, the stable moduli space is nonempty. This is the standard nonemptiness theorem [Yos01; Huy16]. Here v-generic means that semistability for this vector equals stability. For $v = mv_0$ with $m > 0$, the useful nonprimitive consequence is the following: if a semistable sheaf E_0 of vector v_0 exists, then $E_0^{\oplus m}$ is semistable of vector v, since its summands have the same reduced Hilbert polynomial. This does not assert smoothness or stability of the resulting moduli space.

The positivity condition cannot be replaced by "$r > 0$ or l ample": for an ample H, the vector $(-1,H,0)$ has square $H^2 > 0$, but cannot be the vector of a sheaf, whose rank is nonnegative. In positive characteristic we use the moduli theorem with positive rank prime to p from [LO14, Section 3.15], rather than inferring it merely by specializing a complex moduli space.

The stable moduli stack is a $\mathbb{G}_{\mathrm{m}}$-gerbe over $M_H(v)^s$. An algebraic vector w with $\langle v,w\rangle = 1$ trivializes its obstruction to a universal family. Indeed determinant of cohomology against a representative of w has scalar weight $\chi(E,w) = -1$. Tensoring the tautological family by this weight -1 line gives weight zero and therefore descends to the coarse space. In general the tautological family descends only as a twisted sheaf for the Brauer class of the gerbe.

Proposition 3.1.1. *Suppose $v^2 = 0$, $M_H(v) = M_H(v)^s$ is nonempty and fine, and the stable sheaf moduli theorem applies. Its universal family $\mathcal{E}$ defines an equivalence $D^b(M_H(v)) \simeq D^b(X)$.*

Proof. Write $M = M_H(v)$. The preceding deformation calculation makes M a smooth projective surface. Its symplectic form trivializes ω_M. For distinct closed points t,u, stability gives $\mathrm{Hom}(\mathcal{E}_t,\mathcal{E}_u) = 0$ in both directions. Serre duality gives vanishing in degree two, and $\chi(\mathcal{E}_t,\mathcal{E}_u) = -v^2 = 0$ gives degree one. All other degrees vanish because the fibers are sheaves on a surface.

For $t = u$, the map on Ext^0 is the identity on scalars. The map

$$\mathrm{Ext}^1_M(k(t), k(t)) = T_t M \longrightarrow \mathrm{Ext}^1_X(\mathcal{E}_t, \mathcal{E}_t)$$

is the Kodaira–Spencer map of the universal family, hence an isomorphism. It also implies that the map on Ext^2 is nonzero. Choose two degree-one classes whose images have nonzero Yoneda product, possible by the nondegenerate Serre pairing on the two-dimensional target. The functor preserves their product. Both degree-two spaces are one-dimensional, so this map is an isomorphism as well. The spanning-class criterion now gives full faithfulness; equality of dimensions alone would not have done so.

The image is admissible. Since $S_X = [2]$, its semiorthogonal complement is orthogonal in both directions. A connected smooth projective variety has no nontrivial orthogonal decomposition of its derived category: each skyscraper and $\mathcal{O}_X$, being indecomposable, lies in one summand, and the nonzero maps $\mathcal{O}_X \to k(x)$ put them all in that same summand. The skyscrapers form a spanning class. Thus the nonzero image is all of $D^b(X)$. The K3-invariance argument of the preceding chapter shows that M is a K3 surface. □

3.2 Normalizing and lifting an equivalence

Let $\Phi\colon D^b(Y) \to D^b(X)$ preserve the point vector up to sign. A shift makes its action on $(0, 0, 1)$ the identity. Since $(1, 0, 0)$ is isotropic and pairs to -1 with the point vector, its image is $(1, b, b^2/2)$. Tensoring by a line bundle of class $-b$ fixes both outer summands. The orthogonal complement is NS, so the normalized equivalence induces an integral isometry of Picard lattices. Tensoring by a class a acts by

$$(r, l, s) \longmapsto (r, l + ra, s + l \cdot a + ra^2/2).$$

Reflections in classes of (-2)-curves are induced by spherical twists in $\mathcal{O}_C(-1)$. The Weyl-group chamber theorem allows us to arrange that ample cones agree up to sign.

We use the normalized deformation theorem [LO14, Proposition 6.3]: these normalizations give an isomorphism of deformation functors carrying the line-bundle lifting loci into one another, and compatible lifts of the kernel on polarized Artinian deformations. Fixing the point vector alone, before normalization, does not justify the line-bundle formula in that theorem.

Theorem 3.2.1. *A Fourier–Mukai equivalence between K3 surfaces over an algebraically closed field of characteristic different from two which sends* $(0,0,1)$ *to* $\pm(0,0,1)$ *implies* $X \simeq Y$.

Proof. In characteristic zero, the normalization gives an integral Hodge isometry on H^2; the ample-cone adjustment and global Torelli give the result. Descent to a finitely generated field and an embedding into $\mathbb{C}$ establish the same existence statement over any algebraically closed characteristic-zero field.

In positive characteristic choose an ample class on X whose image or negative image is ample on Y. The polarized lifting theorem gives a lift over a finite extension R of $W(k)$. The normalized deformation theorem produces compatible polarized lifts of Y and kernels over the Artinian quotients of R. Grothendieck existence for polarized schemes and perfect complexes algebraizes this compatible system. The cones of the unit and counit for the lifted transform vanish on the special fiber. Derived Nakayama and properness show they vanish on the whole family, so the generic transform is an equivalence. After a finite extension of the fraction field, the characteristic-zero argument gives an isomorphism of generic fibers. The specialization theorem for smooth K3 surfaces over a DVR then identifies the special fibers [LO14, Theorem 6.1]. This last specialization is the K3 version of the Matsusaka–Mumford argument: one obtains a birational map on the special fibers and uses uniqueness of the minimal smooth surface model. No unproved Galois descent of an arbitrary geometric isomorphism is used. □

For a general equivalence put $v = \Phi^N(0,0,1) = (r,l,s)$. It is primitive and isotropic, and has an algebraic pairing partner of pairing one. In characteristic $p > 2$, one can make r positive and prime to p. If r or s is prime to p, use a shift and the spherical twist in $\mathcal{O}_X$, which exchanges r, s up to sign. Otherwise a pairing partner (r', a, s') gives $a \cdot l \equiv 1 \pmod{p}$; tensoring by a changes the last entry to $s + a \cdot l + ra^2/2$, which is prime to p. After the exchange and a shift, $r > 0$. Tensoring by a sufficiently positive ample line bundle makes l very ample. This completes the rank step missing from the original reduction by a shift alone.

The moduli input for this normalized vector gives a nonempty fine K3 moduli space $M_X(v)$. Its equivalence sends the point vector to v. Composing its inverse with Φ fixes the point vector, so the preceding theorem gives $Y \simeq M_X(v)$. Thus every partner has the claimed fine-moduli description. The moduli existence used here is the one in [LO14, Sections 3.15 and 8].

3.3 Partner counts

A Fourier–Mukai partner of X is an isomorphism class of smooth projective varieties with derived category equivalent to $D^b(X)$. Over $\mathbb{C}$ there are finitely many partners. The following sufficient conditions for uniqueness from the original notes are retained: $\rho(X) \geq 12$; or $\rho(X) \geq 3$ with square-free Picard discriminant; or an elliptic fibration with a section. These are quoted lattice-theoretic results [Huy16, Chapter 16].

For a finite-height K3 surface in characteristic $p > 2$, a Picard-preserving projective lift permits the Mukai vectors, generic polarizations and fine moduli spaces to lift. Each special-fiber partner therefore comes from a geometric generic-fiber partner. If two such generic partners are isomorphic, spread their isomorphism over a finite extension of the DVR and specialize as above. Hence the number of special partners is at most the number of generic partners. Picard rank, rather than height, is what is retained in this comparison: height in characteristic p is not defined for the characteristic-zero generic fiber. The Picard-preserving lift and extension of moduli spaces are quoted inputs [LM19; LO14]. For all K3 surfaces in characteristic $p > 2$, including the supersingular case, finiteness and uniqueness when $\rho \geq 11$ are the stronger statements of [LO14, Theorem 1.1]. The following calculation explains supersingular uniqueness directly.

3.4 Crystalline Torelli and characteristic subspaces

Assume now that k is algebraically closed of characteristic $p > 2$. A K3 crystal is a free W-module H with Frobenius φ and a perfect symmetric pairing satisfying

$$p^2 H \subset \varphi(H), \qquad \mathrm{rk}_k(\varphi \bmod p) = 1, \qquad \langle \varphi x, \varphi y \rangle = p^2 \sigma \langle x, y \rangle.$$

It is supersingular when all slopes are one. Its Tate module is

$$T_H = \{x \in H \mid \varphi(x) = px\}.$$

This is a $\mathbb{Z}_p$-module, and supersingularity is equivalent to $T_H \otimes_{\mathbb{Z}_p} W \to H$ being an isogeny. The crystalline Chern class of a line bundle lies in T_H, since $F^* L = L^{\otimes p}$. A marking is an integral lattice $N \hookrightarrow H$ inducing $N \otimes \mathbb{Z}_p \simeq T_H$. For a supersingular K3 surface it is supplied by $\mathrm{Pic}(X)$.

Crystalline Torelli says that an isomorphism of these marked crystals carrying an ample class to an ample class comes from a unique surface isomorphism. The associated existence version says that two supersingular K3 surfaces are isomorphic exactly when their unmarked H^2 crystals are isomorphic. These are quoted Torelli inputs [Ogu79; Bra21]; forgetting the marking asserts existence, not that every unmarked isometry is geometric.

Let $T = T_H$, and let σ_0 be its Artin invariant. The discriminant group $T_0 = T^\vee/T$ is an $\mathbb{F}_p$-space of dimension $2\sigma_0$. Its bilinear form is

$$(\bar{v}, \bar{w}) \longmapsto p^{-1}\langle pv, pw\rangle \bmod p, \qquad v, w \in T^\vee.$$

Changing v by an element of T changes this value by a multiple of p, so the formula is well-defined. With $\varphi = \mathrm{id} \otimes \sigma$ on $T_0 \otimes k$, a strictly characteristic subspace K satisfies

$$\dim K = \sigma_0, \quad \langle K, K\rangle = 0, \quad \dim(K + \varphi K) = \sigma_0 + 1, \quad \sum_{i\geq 0} \varphi^i K = T_0 \otimes k.$$

The inclusions

$$T \otimes_{\mathbb{Z}_p} W \subset H \subset T^\vee \otimes_{\mathbb{Z}_p} W$$

give such a subspace as the image of H in $T_0 \otimes k$. We use this image convention, as in [Bra21, Section 4.1]; Ogus's inverse-Frobenius convention gives an equivalent datum over this perfect field. Isomorphisms of characteristic data lift to isomorphisms of supersingular K3 crystals of the same rank, by the classification theorem. Combining it with crystalline Torelli gives the surface existence criterion in terms of (K, T_0).

Theorem 3.4.1. *Two supersingular K3 surfaces over k are derived equivalent if and only if they are isomorphic.*

Proof. The integral Mukai crystal is $\widetilde{H}(X) = U_W \oplus H^2_{crys}(X/W)$, where the unimodular hyperbolic plane has Frobenius $p\sigma$. This convention is essential: on a Mukai triple the Frobenius is $(a, b, c) \mapsto (p\sigma a, \varphi b, p\sigma c)$. The Tate module is $U_{\mathbb{Z}_p} \oplus T(X)$; unimodularity gives

$$\widetilde{T}(X)^\vee/\widetilde{T}(X) = T(X)^\vee/T(X).$$

The image defining the characteristic subspace is unchanged, because the added summand already equals its Tate lattice tensored with W. A derived equivalence

induces an isomorphism of integral Mukai crystals [Bra21, Theorem 3.5.3]. It therefore identifies the characteristic data of the rank-22 crystals. Their classification and Torelli give $X \simeq Y$. Conversely an isomorphism induces a derived equivalence by pullback. □

For twisted K3 surfaces one uses B-field modified Mukai crystals; the untwisted cancellation above does not assert that a Brauer twist is irrelevant. Nor do the algebraically closed arguments supply automatic descent over an arbitrary perfect field.

Chapter 4

Fractional Calabi–Yau categories

We retain the spherical-functor convention of [Kuz17]. A category is fractional Calabi–Yau if some positive power of its Serre functor is a shift.

4.1 Adjunctions and Lefschetz decompositions

Let $\Phi\colon D^b(X) \to D^b(Y)$ have adjoints $\Phi_L \dashv \Phi \dashv \Phi_R$. Sphericity can be expressed by requiring the adjunction-induced maps

$$\Phi_L \oplus \Phi_R \longrightarrow \Phi_R\Phi\Phi_L, \qquad \Phi_L\Phi\Phi_R \longrightarrow \Phi_L \oplus \Phi_R$$

to be isomorphisms. With the convention in the source, the twists are defined by

$$\Phi_L\Phi \xrightarrow{\epsilon} \mathrm{id} \longrightarrow T_X, \qquad T_Y \longrightarrow \mathrm{id} \xrightarrow{\eta} \Phi\Phi_L.$$

These arrows are precisely the counit and unit of $\Phi_L \dashv \Phi$. In particular, the original directions were correct. Sphericity makes both twists autoequivalences [Kuz17, Proposition 2.13].

A Lefschetz decomposition with respect to a line bundle L_Y is

$$D^b(Y) = \langle \mathcal{B}_0, \mathcal{B}_1 \otimes L_Y, \dots, \mathcal{B}_{m-1} \otimes L_Y^{m-1} \rangle, \qquad \mathcal{B}_0 \supset \cdots \supset \mathcal{B}_{m-1}.$$

It is rectangular if all its blocks equal $\mathcal{B}$. Write L_X, L_Y also for tensoring by the corresponding line bundles. The spherical Lefschetz theorem assumes

$$T_Y(\mathcal{B} \otimes L_Y^i) = \mathcal{B} \otimes L_Y^{i-d}, \qquad L_Y\Phi \simeq \Phi L_X, \qquad T_X L_X \simeq L_X T_X,$$

where $1 \leq d < m$. Its conclusion is

$$D^b(X) = \langle \mathcal{A}_X, \mathcal{B}_X, \mathcal{B}_X \otimes L_X, \ldots, \mathcal{B}_X \otimes L_X^{m-d-1} \rangle, \qquad \mathcal{B}_X = \Phi_L(\mathcal{B}),$$

with Φ_L fully faithful on $\mathcal{B}$. Put $\rho = T_X L_X^d$, $\sigma = S_X T_X L_X^m$, and $c = \gcd(d, m)$. Then

$$S_{\mathcal{A}_X}^{d/c} \simeq \rho^{-m/c} \sigma^{d/c}.$$

This is quoted from [Kuz17, Theorem 3.5]; the following substitutions are the explicit calculations used here.

4.2 Hypersurfaces

Let $i\colon X \hookrightarrow Y$ be a smooth divisor of degree d in a smooth variety of dimension N, with $\omega_Y = L_Y^{-m}$, and let $L_X = i^* L_Y$. For $\Phi = i_*$, the left adjoint is Li^*. The divisor resolution and its self-intersection triangle give

$$\begin{aligned} T_Y &= L_Y^{-d}, & T_X &= L_X^{-d}[2], \\ \rho &= [2], & \sigma &= [N+1]. \end{aligned}$$

For the second equality in the first line, the fiber of $Li^* i_* F \to F$ is $F \otimes L_X^{-d}[1]$; its cofiber is therefore $F \otimes L_X^{-d}[2]$. Adjunction gives $S_X = L_X^{d-m}[N-1]$, which yields the expression for σ. The theorem consequently reads

$$S_{\mathcal{A}_X}^{d/c} = [((N+1)d - 2m)/c].$$

When $d = m$, take $\mathcal{A}_X = D^b(X)$; ordinary adjunction gives the same formula, without asserting full faithfulness of a Lefschetz block.

For $Y = \mathbb{P}^n$, the exceptional line bundles give a rectangular decomposition of length $m = n + 1$. Thus

$$D^b(X) = \langle Ku(X), \mathcal{O}_X, \ldots, \mathcal{O}_X(n-d) \rangle, \qquad S_{Ku(X)}^{d/c} = [(n+1)(d-2)/c].$$

For a cubic fourfold $n = 5, d = 3, c = 3$, and $S_{Ku(X)} = [2]$. This is the K3-category property used in the next chapters.

For $Y = \mathrm{Gr}(k, n)$ with $\gcd(k, n) = 1$, use the rectangular decomposition of length n, whose first block is generated by the Schur functors in the standard exceptional collection. Its dimension is $N = k(n-k)$ and $\omega_Y = L_Y^{-n}$. For a degree-d hypersurface, $d \leq n$, the result is

$$S_{Ku(X)}^{d/c} = [((k(n-k)+1)d - 2n)/c], \qquad c = \gcd(d, n).$$

The blocks following $Ku(X)$ are $\mathcal{B}_X, \ldots, \mathcal{B}_X(n-d-1)$. If $d \mid n$, the component is Calabi–Yau of dimension $k(n-k)+1-2n/d$. The integer entering the greatest common divisor is the Lefschetz length n; using $n+1$ in this Grassmannian formula is incorrect.

Chapter 5

Stability on the K3 category of a cubic fourfold

Throughout this chapter, X is a smooth complex cubic fourfold unless otherwise stated. The construction and results below are quoted from [Kuz17; BAY+23]; this chapter is an account of that construction. The derived category of a Fano variety X has a special subcategory named Kuznetsov component $Ku(X)$. In a series of papers, Kuznetsov has shown that the geometry of Fano varieties and their moduli spaces can be captured efficiently by $Ku(X)$. We will focus on the cubic fourfold, in this case, $Ku(X)$ is defined by the semiorthogonal decomposition

$$D^b(X) = \langle Ku(X), \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle$$

where H is a hyperplane section. Here $Ku(X)$ is a K3 category (i.e. its Serre functor is $[2]$). It is a long-standing conjecture that $Ku(X)$ is the derived category of a K3 surface if and only if X is rational. No rationality criterion or proof of this conjectural relation is established in these notes.

For a smooth complex cubic fourfold, $Ku(X)$ has a Bridgeland stability condition [BAY+23]. So one can hope to recover X or some interesting information from $Ku(X)$ and its stability condition.

The preceding chapter supplies the fractional Calabi–Yau calculation.

5.1 Bridgeland stability

This construction of stability condition on Kuznetsov component mainly follows [BAY+23].

Definition 5.1.1. *A heart of a bounded t-structure in a triangulated category $\mathcal{D}$ is a full subcategory $\mathcal{C} \subset \mathcal{D}$ such that*

1. *for $E, F \in \mathcal{C}$ and $n < 0$ we have $\mathrm{Hom}(E, F[n]) = 0$,*
2. *for every $E \in \mathcal{C}$ there exists a sequence of morphism*

$$0 = E_0 \xrightarrow{\varphi_1} E_1 \cdots \xrightarrow{\varphi_m} E_m = E$$

such that the cofiber of φ_i is of the form $A_i[k_i]$ for some sequence $k_1 > k_2 > \cdots > k_m$ and $A_i \in \mathcal{C}$.

Remark 5.1.2. *It is easy to see this definition uniquely determines the bounded t-structure, we use this definition just for convenience.*

Definition 5.1.3. *Let Λ be a finite rank free abelian group and let $c\colon K(\mathcal{D}) \to \Lambda$ be a surjective group homomorphism. A Bridgeland stability condition on $\mathcal{D}$ with respect to the pair (Λ, c) is a pair $\sigma = (\mathcal{C}, Z)$ consisting of the heart of a bounded t-structure $\mathcal{C} \subset \mathcal{D}$ and a group homomorphism $Z\colon \Lambda \to \mathbb{C}$ (called central charge) such that:*

1. *For every $0 \neq A \in \mathcal{C}$, $\Im(Z(A)) \geq 0$ and if $\Im Z(A) = 0$ then $\Re(Z(A)) < 0$,*
2. *Define the slope by setting $\mu_\sigma := -\frac{\Re Z}{\Im Z}$. Every nonzero object $A \in \mathcal{C}$ has a finite Harder–Narasimhan filtration with semistable factors of strictly decreasing slope.*
3. *There exists a quadratic form Q on $\Lambda \otimes \mathbb{R}$ such that $Q|_{\ker Z}$ is negative definite, and $Q(E) \geq 0$ for all σ-semistable objects.*

Embedding of the Kuznetsov component Let $X \subset \mathbb{P}^{N+2}$ be a smooth cubic hypersurface of dimension $N+1$ over an algebraically closed field of characteristic $\neq 2$. Let $L_0 \subset X$ be a line not contained in a plane in X, consider the blow up $\sigma\colon \widetilde{X} \to X$ along L_0, and denote by $i\colon D \to \widetilde{X}$ its exceptional divisor. Then the projection from L_0 onto the projective space $\mathbb{P}^N$ induces a conic fibration $\pi\colon \widetilde{X} \to \mathbb{P}^N$.

For cubic fourfold, we get the conic fibration $\pi\colon \widetilde{X} \to \mathbb{P}^3$. There is a rank three vector bundle $\mathcal{F}$ on $\mathbb{P}^3$ and a line bundle $\mathcal{L}$ such that $\widetilde{X}$ embeds into the $\mathbb{P}^2$-bundle $\mathbb{P}_{\mathbb{P}^3}(\mathcal{F})$ as the zero locus of a section

$$s_X \in H^0(\mathbb{P}^3, \operatorname{Sym}^2 \mathcal{F}^\vee \otimes \mathcal{L}^\vee) = H^0(\mathbb{P}_{\mathbb{P}^3}(\mathcal{F}), \mathcal{O}_{\mathbb{P}_{\mathbb{P}^3}(\mathcal{F})}(2) \otimes \mathcal{L}^\vee).$$

In particular, we have an associated sheaf of Clifford algebras over $\mathbb{P}^3$, whose even part (resp. odd part) is denoted by $\mathcal{B}_0$ (resp. $\mathcal{B}_1$). Let h be the hyperplane class on $\mathbb{P}^3$ and we use the same notation for its pullback to $\widetilde{X}$. We consider the $\mathcal{B}_0$-bimodules

$$\mathcal{B}_{2j} := \mathcal{B}_0 \otimes \mathcal{O}_{\mathbb{P}^3}(jh), \mathcal{B}_{2j+1} := \mathcal{B}_1 \otimes \mathcal{O}_{\mathbb{P}^3}(jh)$$

for $j \in \mathbb{Z}$. By [BAY+23, Section 7], we have

$$D^b(\mathbb{P}^3, \mathcal{B}_0) = \langle \Psi(\sigma^* Ku(X)), \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle,$$

where $\Psi \circ \sigma^*$ gives a fully faithful embedding of $Ku(X)$, and $\Psi\colon D^b(\widetilde{X}) \to D^b(\mathbb{P}^3, \mathcal{B}_0)$ is defined by

$$\Psi(-) = \pi_*(- \otimes \mathcal{O}_{\widetilde{X}}(h) \otimes \mathcal{E}[1]).$$

Here $\mathcal{E}$ is a sheaf of right $\pi^*\mathcal{B}_0$-modules on $\widetilde{X}$, and it is a vector bundle of rank 2.

Construction of stability conditions We now construct stability conditions on the category $D^b(\mathbb{P}^3, \mathcal{B}_0)$. We define the twisted Chern character as

$$\operatorname{ch}_{\mathcal{B}_0}(E) := \operatorname{ch}(E)(1 - \frac{11}{32}l)$$

for all $E \in D^b(\mathbb{P}^3, \mathcal{B}_0)$, where l denotes the class of a line in $\mathbb{P}^3$. Moreover, for a given real number β, we consider the modified twisted Chern character $\operatorname{ch}^\beta_{\mathcal{B}_0} := e^{-\beta h} \operatorname{ch}_{\mathcal{B}_0}$. And we denote by $\operatorname{ch}_{\mathcal{B}_0,i}$ the degree i component of $\operatorname{ch}_{\mathcal{B}_0}$.

Write $\operatorname{Coh}(\mathbb{P}^3, \mathcal{B}_0)$ for coherent right $\mathcal{B}_0$-modules. For an object of positive rank, put $\mu_h^{\mathcal{B}_0} = \operatorname{ch}_{\mathcal{B}_0,1} / \operatorname{ch}_{\mathcal{B}_0,0}$, after identifying the first Chern component with its coefficient of h; torsion sheaves have infinite slope. For $\beta \in \mathbb{R}$, let $\mathcal{T}_\beta$ consist of sheaves whose Harder–Narasimhan factors have slopes greater than β, and let $\mathcal{F}_\beta$ consist of those whose factors have slopes at most β. The tilted heart is

$$\operatorname{Coh}^\beta(\mathbb{P}^3, \mathcal{B}_0) = \langle \mathcal{F}_\beta[1], \mathcal{T}_\beta \rangle.$$

The weak stability construction uses the truncated Chern-character lattice $\Lambda_{\mathcal{B}_0}$, the image of the first three Chern components. Throughout the formulas below these components are their coefficients in $1, h, h^2$, so the displayed discriminant is a number.

The key result is the following inequality for slope-semistable sheaves:

Theorem 5.1.4. *For any $\mu_h^{\mathcal{B}_0}$-semistable sheaf $E \in \mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$, we have*

$$\Delta_{\mathcal{B}_0}(E) := \mathrm{ch}_{\mathcal{B}_0,1}(E)^2 - 2\,\mathrm{ch}_{\mathcal{B}_0,0}(E)\,\mathrm{ch}_{\mathcal{B}_0,2}(E) \geq 0.$$

With the correction given by $\frac{11}{32}$, we have $\Delta_{\mathcal{B}_0}(\mathcal{B}_i) = 0$. By the preceding theorem we have a quadratic form $\Delta_{\mathcal{B}_0}$ on $\Lambda_{\mathcal{B}_0}$.

Proposition 5.1.5. *Given $\alpha > 0$ and $\beta \in \mathbb{R}$, the pair*

$$\sigma_{\alpha,\beta} = (\overset{\beta}{\mathrm{Coh}}(\mathbb{P}^3, \mathcal{B}_0), Z_{\alpha,\beta})$$

with

$$Z_{\alpha,\beta}(\mathcal{F}) = i\,\mathrm{ch}^{\beta}_{\mathcal{B}_0,1}(\mathcal{F}) + \frac{1}{2}\alpha^2\,\mathrm{ch}^{\beta}_{\mathcal{B}_0,0}(\mathcal{F}) - \mathrm{ch}^{\beta}_{\mathcal{B}_0,2}(\mathcal{F})$$

defines a weak stability condition on $D^b(\mathbb{P}^3, \mathcal{B}_0)$. The quadratic form can be given by the discriminant $\Delta_{\mathcal{B}_0}$. In particular, for a $\sigma_{\alpha,\beta}$-stable object $\mathcal{F}$, we have

$$\Delta_{\mathcal{B}_0}(\mathcal{F}) \geq 0.$$

Remark 5.1.6. *The proof of this construction can be found in [BAY+23, Section 9].*

We recall that when $\mathrm{ch}^{\beta}_{\mathcal{B}_0,1}(\mathcal{F}) \neq 0$ the slope of $\mathcal{F}$ associated to $\sigma_{\alpha,\beta}$ is defined as

$$\mu_{\alpha,\beta}(\mathcal{F}) = \frac{\mathrm{ch}_{\mathcal{B}_0,2}(\mathcal{F}) - \frac{1}{2}(\alpha^2+\beta^2)\,\mathrm{rank}(\mathcal{F})}{\mathrm{ch}_{\mathcal{B}_0,1}(\mathcal{F}) - \beta\,\mathrm{rank}(\mathcal{F})} - \beta.$$

Theorem 5.1.7. *If X is a cubic fourfold, then there exists a stability condition on $Ku(X)$.*

Proof. We fix $0 < \alpha < \frac{1}{4}$ and $\beta = -1$, and we consider the tilting of $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$ with respect to $\mu_{\alpha,\beta} = 0$. The new heart is denoted by $\mathrm{Coh}^0_{\alpha,-1}(\mathbb{P}^3, \mathcal{B}_0)$. Note that $Ku(X)$ embeds into $D^b(\mathbb{P}^3, \mathcal{B}_0)$, then as shown in [BAY+23, Section 9], the pair

$$\sigma_\alpha = (\mathrm{Coh}^0_{\alpha,-1}(\mathbb{P}^3, \mathcal{B}_0) \cap Ku(X), -iZ_{\alpha,-1})$$

defines a Bridgeland stability condition on $Ku(X)$. □

The next chapter treats the line-independence criterion, stable-object moduli and the categorical and classical Torelli arguments.

Chapter 6

Moduli spaces and Torelli for cubic fourfolds

Throughout this chapter, cubic fourfolds are smooth and complex. We develop the argument of the original notes with the hypotheses used in [BAY+23, Appendix A].

6.1 The Mukai lattice and stability space

The topological Mukai lattice of $Ku(X)$ is

$$\widetilde{H}(Ku(X),\mathbb{Z}) = \{e \in K_{top}(X) \mid \chi(\mathcal{O}_X(i), e) = 0,\ i = 0,1,2\},$$

with pairing $-\chi$. Its algebraic part is the numerical Grothendieck lattice $N(Ku(X))$, of signature $(2,\rho)$. These identifications and the induced Hodge structure are quoted [AT14]. For a line $L \subset X$, put

$$\lambda_1 = [\mathrm{pr}(\mathcal{O}_L(1))], \qquad \lambda_2 = [\mathrm{pr}(\mathcal{O}_L(2))].$$

They span the distinguished primitive positive lattice

$$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Here pr is the projection to the Kuznetsov component; the class $N(Ku(X))$ is distinct from the surface lattice $N(X)$ of the K3 chapter.

A numerical central charge can be written uniquely as $Z(-) = \langle \eta(\sigma), - \rangle$, where $\eta(\sigma) \in N(Ku(X)) \otimes \mathbb{C}$. It generally does not belong to the integral lattice. Let $\mathcal{P}$ be the open set of vectors whose real and imaginary parts span a positive plane, and put

$$\mathcal{P}_0 = \mathcal{P} \setminus \bigcup_{\delta^2=-2} \delta^{\perp}.$$

The central-charge map is locally a homeomorphism. The distinguished component containing the stability conditions constructed in the preceding chapter covers the corresponding component of $\mathcal{P}_0$; this is the cubic-category statement in [BAY+23, Section 9], not a direct application of a theorem only stated for derived categories of K3 surfaces. Those constructed conditions have $\eta(\sigma) \in (A_2)_{\mathbb{C}} \cap \mathcal{P}_0$.

6.2 The line-independence argument

The original proof contains a useful general criterion. Suppose an irreducible parameter space T carries stability conditions σ_t with the same central charge. Suppose further that for every fixed object and every real phase ϕ, the locus $\{t \mid E \in \mathcal{P}_t(\phi)\}$ is open. Then all the slicings are equal.

Indeed, let $E \in \mathcal{P}_{t_0}(\phi)$ and suppose it is not semistable of that phase at t_1. If its largest HN phase exceeds ϕ, take the first HN factor A, with its nonzero map $A \to E$. The two open semistability loci meet by irreducibility, contradicting the vanishing of maps from a larger phase to a smaller phase. If the smallest HN phase is below ϕ, use instead the last factor and $E \to B$, with the same contradiction. If neither occurs, every HN factor has phase ϕ, so E was already semistable of that phase. Applying this for all objects and phases identifies the slicings and hence the hearts.

For the construction from a line in the cubic, work over the nonempty open subset $F^0(X)$ of the irreducible Fano variety parametrizing lines not contained in a plane. The constant cubic family with its universal line gives the relative Clifford construction, and [Bay+21, Proposition 30.5] supplies a family of stability conditions, including openness of semistability with a specified phase. The numerical formula identifies their central charges. Apply the criterion to the constant families of the fixed objects E, A, B. It follows that, for fixed $0 < \alpha < 1/4$, the constructed stability condition is independent of the admissible line. This is [LPZ18, Proposition 2.6], which used the earlier numbering 30.4 for the family input. The proof uses that full input, rather than inferring openness of semistability just from membership in a heart.

6.3 Small Ext groups and stable objects

Write $\operatorname{ext}^i(E,F) = \dim_{\mathbb{C}} \operatorname{Ext}^i(E,F)$. An object is rigid if $\operatorname{ext}^1(E,E) = 0$ and semirigid if this dimension is two. In a K3 category, Mukai's inequality for a triangle $A \to E \to B$ with $\operatorname{Hom}(A,B) = 0$ is

$$\operatorname{ext}^1(A,A) + \operatorname{ext}^1(B,B) \leq \operatorname{ext}^1(E,E).$$

We use this lemma and its stable-factor reduction [BAY+23, Lemmas A.2–A.4]. Applied successively to HN and Jordan–Holder factors, the reduction makes a rigid object semistable with a single stable factor type. It likewise makes a semirigid object semistable with a single stable factor type once nonzero rigid objects are excluded. This is the categorical input to the following numerical calculations, which alone would not give the reduction.

Assume that $N(Ku(X))$ has no classes of square -2. If a nonzero rigid object reduces to a semistable object E with unique stable factor type E_0, then $v(E) = mv(E_0)$ for some $m > 0$. Serre duality and semistability give $v(E)^2 = -2\hom(E,E) < 0$. But stability of E_0 gives $v(E_0)^2 = \operatorname{ext}^1(E_0,E_0) - 2$. Its degree-one Ext group is symplectic and therefore even-dimensional; a negative square would have to be -2, a contradiction.

For a semirigid object the same reduction gives $v(E)^2 = 2 - 2\hom(E,E) \leq 0$. Since $v(E_0)^2 \geq 0$, both squares are zero and $\operatorname{hom}(E,E) = 1$. If $m > 1$, a simple quotient $E \twoheadrightarrow E_0$ followed by a simple subobject $E_0 \hookrightarrow E$ gives a nonzero noninvertible endomorphism. This contradicts simplicity, so $m = 1$ and E is stable.

Now suppose additionally that

$$\operatorname{Ext}^{<0}(E,E) = 0, \qquad \operatorname{Hom}(E,E) = \mathbb{C}, \qquad \operatorname{ext}^1(E,E) = 4.$$

The negative-Ext hypothesis and Serre duality ensure that $v(E)^2 = 2$. This vector is primitive, since its square is two in an even lattice. If E is not stable, the same factor reduction and Mukai's inequality leave precisely two semirigid stable pieces A, B. Their squares are zero, and

$$(v(A) + v(B))^2 = 2, \qquad (v(A) - v(B))^2 = -2,$$

contradicting the assumption. This is the proof of [BAY+23, Lemma A.5]; the original statement omitted the negative-Ext hypothesis needed to calculate the Euler characteristic.

6.4 Lines as a moduli space

For a line $L \subset X$, the evaluation sequence defines

$$0 \longrightarrow F_L \longrightarrow H^0(X, I_L(1)) \otimes \mathcal{O}_X \longrightarrow I_L(1) \longrightarrow 0.$$

The evaluation is surjective, since the linear equations of L generate its ideal. Thus F_L has rank three and equals $\mathbb{L}_{\mathcal{O}_X}(I_L(1))[-1]$. The construction in [KM07] gives a reflexive sheaf in $Ku(X)$, with $v(F_L) = \lambda_1$ and

$$\hom(F_L, F_L) = 1, \quad \operatorname{ext}^1(F_L, F_L) = 4, \quad \operatorname{ext}^2(F_L, F_L) = 1.$$

In particular the dimension four belongs to Ext^1, not Ext^2. When there are no (-2)-classes, every F_L is stable for every numerical stability condition.

For a fixed phase and numerical vector, use the moduli space of stable objects in that phase. Fixing a phase, or equivalently choosing its representatives in a fixed heart, removes the infinite copies obtained by even shifts. The original notation with $\pm v$ is interpreted with this phase normalization. The Fano fourfold $F(X)$, with its universal family F_L, gives a proper smooth component of dimension $\lambda_1^2 + 2 = 4$. The completeness theorem for this component [BAY+23, Proposition A.7 and Theorem A.8] identifies it with the entire phase-normalized stable moduli space of class λ_1.

For $\eta(\sigma) \in (A_2)_{\mathbb{C}} \cap \mathcal{P}$, the moduli nef divisor is a positive multiple of the Plucker class. Under the Mukai identification of the Picard lattice with $\lambda_1^{\perp}$, its direction is $\lambda_1 + 2\lambda_2$, as follows from

$$\langle \lambda_1, \lambda_1 + 2\lambda_2 \rangle = 0, \qquad (\lambda_1 + 2\lambda_2)^2 = 6.$$

The identification with the actual Plucker polarization is a geometric input of the cited theorem, not a consequence of the square alone.

6.5 Categorical Torelli and classical Torelli

Let (1) denote the degree shift, namely tensoring by $\mathcal{O}_X(1)$ followed by projection to $Ku(X)$. Its action cyclically permutes $\lambda_1, \lambda_2, -\lambda_1 - \lambda_2$ and is the identity on $A_2^{\perp}$. In particular its nontrivial eigenspace determines A_2 rationally, and primitivity determines the integral distinguished sublattice.

Assume $N(Ku(X))$ has no (-2)-classes. The categorical Torelli theorem used here says that $X \simeq Y$ if an equivalence $\Phi\colon Ku(X) \to Ku(Y)$ commutes

with this action on the algebraic Mukai lattice [BAY+23, Theorem A.1]. The original stronger statement for arbitrary cubics is not established by the argument with small Ext groups, whose hypothesis must be retained.

Here is the moduli argument. The equivalence preserves A_2, and composing with degree shifts and shifts normalizes its action to fix λ_1, λ_2. Transport a stability condition σ and its stable objects to $\Phi(\sigma)$. If the extension $D^b(X) \to Ku(X) \xrightarrow{\Phi} Ku(Y) \to D^b(Y)$ is Fourier–Mukai, apply its kernel on the first factor of the universal family on $X \times F(X)$. The result is a family on $Y \times F(X)$, over the same parameter scheme $F(X)$. Its classifying morphism is $F(X) \to F(Y)$. Applying the inverse kernel gives the inverse morphism, so this is an isomorphism. Transport of central charges and determinant line bundles preserves the nef divisor, hence the Plucker polarization. The polarized Fano-variety reconstruction gives $X \simeq Y$ [Cha12]. For an arbitrary exact equivalence, the construction of the transported specific family is an additional representability argument; it is included in the cited Theorem A.1. One must not write an unconstructed product functor or replace the parameter space before classifying the family. Conversely an isomorphism of cubics supplies the required equivalence and compatibility.

The classical Torelli theorem states that two smooth complex cubic fourfolds are isomorphic if and only if their primitive fourth integral cohomologies are Hodge isometric. The deformation proof in the original notes is the proof of [BAY+23, Theorem A.9]. Its precise external input extends the primitive isometry to an oriented Mukai isometry over local deformation spaces, and supplies a dense set of pairs whose Kuznetsov components are twisted K3 categories with no (-2)-classes. On this set the Hodge isometries lift to equivalences compatible with the degree-shift action. The categorical theorem just proved gives isomorphic cubic fibers. The two local maps to the separated moduli space of cubics agree on a dense set, hence everywhere: their equalizer is closed, being the inverse image of the diagonal. Thus the original fibers agree. Density by itself would not give the degree-shift compatibility; both are part of the cited deformation input. This completes the dependency chain without removing the root-free hypothesis at the intermediate categorical step.

Appendix A

F-crystals and height

For a perfect field k of positive characteristic p, $W := W(k)$ is its Witt ring, and we denote by K its field of fraction. The Frobenius map on W is denoted by σ.

Definition A.0.1 (F-(iso)crystal)**.** *An F-crystal (M, φ_M) over k is a free W-module M of finite rank with an injective and σ-linear map $\varphi_M \colon M \to M$. Specifically, φ_M is additive, injective and satisfies*

$$\varphi_M(r \cdot m) = \sigma(r) \cdot \varphi_M(m) \text{ for all } r \in W, m \in M.$$

An F-isocrystal (V, φ_V) is a finite dimensional K-vector space V together with an injective and σ-linear map $\varphi_V \colon V \to V$.

A morphism $u \colon (M, \varphi_M) \to (N, \varphi_N)$ of F-crystals (F-isocrystals) is a W-linear (K-linear) map $u \colon M \to N$ such that $\varphi_N \circ u = u \circ \varphi_M$. An isogeny of F-crystals is a morphism $u \colon (M, \varphi_M) \to (N, \varphi_N)$, such that the induced map $M \otimes_W K \to N \otimes_W K$ is an isomorphism of F-isocrystals.

Example A.0.2. *Let X be a smooth and proper variety over k. Then $H^n := H^n_{cris}(X/W)/\text{torsion}$ is a free W-module of finite rank. The Frobenius induces a σ-linear map $\varphi \colon H^n \to H^n$, and the Poincare duality induces a perfect pairing*

$$\langle -, - \rangle \colon H^n \times H^{2 \dim X - n} \to H^{2 \dim X}(X/W) \simeq W,$$

which is compatible with Frobenius

$$\langle \varphi(x), \varphi(y) \rangle = p^{\dim X} \cdot \sigma(\langle x, y \rangle).$$

So $\varphi \colon H^n \to H^n$ is injective and thus (H^n, φ) is an F-crystal.

Example A.0.3. *Let $W_\sigma\langle T\rangle$ be the non-commutative polynomial ring in the variable T over W subject to the relations $T \cdot r = \sigma(r) \cdot T$ for $r \in W$.*

Let $\alpha = r/s \in \mathbb{Q}_{\geq 0}$, where $r \geq 0$, $s > 0$, and r, s are coprime integers. Then

$$M_\alpha := W_\sigma\langle T\rangle / W_\sigma\langle T\rangle (T^s - p^r)$$

together with $\varphi\colon m \mapsto T \cdot m$ defines an F-crystal (M_α, φ) of rank s. The number α is called the slope of (M_α, φ).

Theorem A.0.4. *Let k be an algebraically closed field of positive characteristic. Then the category of F-crystals over k up to isogeny is semisimple, and the simple objects are of the form $(M_\alpha, \varphi), \alpha \in \mathbb{Q}_{\geq 0}$.*

Definition A.0.5. *Let (M, φ) be an F-crystal over an algebraically closed field k of positive characteristic, and let $M \simeq \bigoplus_{\alpha \in \mathbb{Q}_{\geq 0}} M_\alpha^{n_\alpha}$ be its decomposition up to isogeny.*

Then the elements in the set $\{\alpha \in \mathbb{Q}_{\geq 0}, n_\alpha \neq 0\}$ are called the slopes of (M, φ).

For every slope α the integer $\lambda_\alpha := n_\alpha \cdot \mathrm{rk}_W M_\alpha$ is called the multiplicity of the slope α.

When (M, φ) is an F-crystal over a perfect field k, then its slopes and multiplicities are defined as the ones of $(M, \varphi) \otimes_{W(k)} W(\bar{k})$, where $\bar{k}$ is an algebraic closure of k.

Definition A.0.6. *Let X be a K3 surface over a perfect field of characteristic $p > 2$. If X is not supersingular, its height is*

$$h(X) = \dim_K H^2_{crys}(X/K)_{[0,1)}.$$

If X is supersingular, its height is ∞, not zero.

For an F-crystal (M, φ). We order its slopes in ascending order

$$0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_t$$

and denote by $\lambda_1, \cdots, \lambda_t$ the respective multiplicities.

Then the Newton polygon is defined to be the graph of the piecewise linear function Nwt_M from the interval $[0, \mathrm{rk}_W M]$ to $\mathbb{R}$, such that $\mathrm{Nwt}_M(0) = 0$ and has slopes α_i when $x \in [\lambda_1 + \cdots + \lambda_{i-1}, \lambda_1 + \cdots + \lambda_{i-1} + \lambda_i]$. Since we ordered the slopes in ascending order, this polygon is convex.

Next we define the Hodge polygon of (M, φ). We have an isomorphism $M/\varphi(M) \simeq \bigoplus_{i\geq 1}(W/p^iW)^{h_i}$ for some non-negative integers h_i. We define $h_0 := \operatorname{rk} M - \sum_i h_i$ and call them as Hodge numbers. For a smooth and proper scheme X over k, if the Hodge to de Rham spectral sequence of X degenerates at E_1 and $H^n_{crys}(X/W)$ is torsion free, then $h_i(H^{i+j}_{crys}(X/W)) = \operatorname{rk}_k H^j(X, \Omega^i_X)$, i.e. the Hodge numbers coincide with the classical Hodge numbers.

Then the Hodge polygon is defined to be the graph of the piecewise linear function Hdg_M from the interval $[0, \operatorname{rk}_W M]$ to $\mathbb{R}$, such that $\mathrm{Hdg}_M(0) = 0$ and has slopes i when $x \in [h_0 + \cdots + h_{i-1}, h_0 + \cdots + h_{i-1} + h_i]$. This polygon is convex, too.

Proposition A.0.7. *Let (M, φ) be an F-crystal. the its Newton-polygon lies on or above its Hodge polygon, and both have same startpoint and endpoint.*

The preceding inequality concerns the algebraic Hodge polygon of the crystal. For a smooth proper scheme whose crystalline cohomology is torsion-free and whose Hodge-to-de Rham spectral sequence degenerates, the crystalline Hodge comparison identifies these elementary divisors with the geometric Hodge numbers. Under those hypotheses the same Newton–Hodge inequality therefore applies to the geometric Hodge polygon. For a smooth proper lift to W, the filtered crystalline–de Rham comparison supplies the Newton–Hodge inequality for its characteristic-zero generic fiber. These geometric comparisons are quoted inputs; they do not follow merely from the elementary-divisor definition. The K3 hypotheses required here were checked in the cohomological chapter. The slope description for K3 surfaces is the following standard result [Huy16].

Theorem A.0.8. *For a K3 surface, if X is not supersingular then there exists an integer $h \in \{1, 2, \cdots, 10\}$ such the Newton polygon of $H^2_{crys}(X/W)$ has slopes*

$$(1 - 1/h, 1, 1 + 1/h)$$

with multiplicities

$$(h, 22 - 2h, h).$$

This number h is the height of formal Brauer group, for supersingular K3 surface, we set their height as ∞.

Appendix B

Deformations of K3 surfaces

Let X be a K3 surface over an algebraically closed field k of characteristic $p > 2$. Its tangent and obstruction spaces are $H^1(X, T_X)$ and $H^2(X, T_X)$. Contraction with a nonzero two-form identifies T_X with Ω^1_X. Consequently

$$H^0(X, T_X) = H^2(X, T_X) = 0, \qquad \dim H^1(X, T_X) = 20.$$

The standard infinitesimal deformation theory therefore gives a noncanonical identification

$$\mathrm{Def}_X \simeq \mathrm{Spf}\, W(k)[[t_1, \ldots, t_{20}]].$$

Here the obstruction vanishing supplies formal smoothness over $W(k)$; the tangent-space calculation supplies relative dimension twenty.

B.1 Preserving a line bundle

For a line bundle L, the Atiyah extension is

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{D}(L) \longrightarrow T_X \longrightarrow 0.$$

Its extension class is $c_1^H(L) \in H^1(X, \Omega^1_X)$. The deformation theory of (X, L) is governed by $\mathcal{D}(L)$. Since $H^1(\mathcal{O}_X) = H^2(T_X) = 0$, its cohomology sequence contains

$$0 \longrightarrow H^1(\mathcal{D}(L)) \longrightarrow H^1(T_X) \xrightarrow{\cup c_1^H(L)} H^2(\mathcal{O}_X) \longrightarrow H^2(\mathcal{D}(L)) \longrightarrow 0.$$

Serre duality identifies the middle arrow with the functional defined by $c_1^H(L)$. If this class is nonzero, the arrow is surjective, the obstruction space vanishes and

the tangent space has dimension nineteen. Thus $\mathrm{Def}_{X,L}$ is formally smooth over $W(k)$ of relative dimension nineteen. The nonzero class also means that the local equation for preserving L has nonzero differential, so it defines a smooth Cartier divisor inside Def_X.

This calculation disproves the unconditional smoothness assertion for every nontrivial L in the draft. If $L = M^{\otimes p}$ with M ample, then L is nontrivial but $c_1^H(L) = pc_1^H(M) = 0$. Every first-order deformation of X preserves L, so the tangent space has dimension twenty, incompatible with smooth relative dimension nineteen. Vanishing of this linear obstruction does not mean that all higher obstructions vanish. The flat Cartier-divisor theorem for nontrivial line bundles requires its full deformation-theoretic input [Dl81]; it cannot be strengthened to smoothness from nontriviality alone.

B.2 Projective lifting

The polarized lifting theorem used in these notes asserts existence of a projective lift of an ample pair after a finite extension of the Witt ring. We use it in this form, as in [LO14, proof of Theorem 6.1]. When $c_1^H(L) \neq 0$, the formally smooth deformation space gives a compatible lift of the pair over $W(k)$ itself. Formal existence for the ample line bundle then algebraizes the formal surface.

The original notes also singled out a superspecial Kummer calculation. Here is the projective lifting calculation in its precise scope. Let $X = \mathrm{Km}(E \times E)$ for an elliptic curve E/k. Lift a Weierstrass equation with invertible discriminant to $W(k)$ and call the resulting elliptic scheme $\mathcal{E}$. The sixteen two-torsion points of $\mathcal{E} \times \mathcal{E}$ form a finite etale subgroup scheme because 2 is invertible. Blow up that subgroup and take the quotient by the lifted involution $[-1]$. Locally at a fixed point the blow-up has coordinates (x, t), with involution $(x, t) \mapsto (-x, t)$, so its quotient has regular coordinates (x^2, t). Away from the exceptional divisors the involution acts freely. The quotient is therefore a smooth projective relative surface, and its special fiber is X. This proves existence of a projective $W(k)$-lift for this Kummer surface. It does not assert that every chosen polarization on it lifts to that model. The identification of the Artin-invariant-one supersingular K3 surface with the appropriate Kummer surface is a separate quoted classification input [Ogu79]; it is not a consequence of the Tate conjecture alone.

Appendix C

Etale cohomology calculations

Let k be algebraically closed. In the prime-to-characteristic calculation let n be invertible in k, and put $\Lambda = \mathbb{Z}/n\mathbb{Z}$. Tate twists are retained until a choice $\mu_n \simeq \Lambda$ is made.

C.1 Kummer sequence and curves

For a smooth projective connected curve C of genus g, the Kummer sequence and $\mathrm{Br}(C) = 0$ give

$$H^1_{et}(C, \mu_n) \simeq \mathrm{Pic}(C)[n],$$
$$H^2_{et}(C, \mu_n) \simeq \mathrm{Pic}(C)/n\,\mathrm{Pic}(C).$$

The first isomorphism uses surjectivity of $k^* \xrightarrow{n} k^*$. Apply the snake lemma to multiplication by n on

$$0 \longrightarrow \mathrm{Pic}^0(C) \longrightarrow \mathrm{Pic}(C) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

Multiplication by n on the Jacobian is surjective with kernel $(\mathbb{Z}/n\mathbb{Z})^{2g}$. Consequently

$$\mathrm{Pic}(C)[n] = \mathrm{Pic}^0(C)[n], \qquad \mathrm{Pic}(C)/n\,\mathrm{Pic}(C) = \mathbb{Z}/n\mathbb{Z}.$$

Together with connectedness and the etale cohomological dimension bound for curves, this gives

$$H^q_{et}(C,\Lambda) \simeq \begin{cases} \Lambda & q=0, \\ \Lambda^{2g} & q=1, \\ \Lambda(-1) & q=2, \\ 0 & q>2. \end{cases}$$

The Brauer vanishing and cohomological dimension bound are separate inputs; vanishing of higher cohomology does not follow from the Picard calculation alone.

C.2 Purity, affine space and projective space

For a smooth closed subvariety $i\colon Z \hookrightarrow X$ of codimension c, with smooth ambient X and complement U, purity identifies $i^!\Lambda$ with $\Lambda(-c)[-2c]$. The localization triangle therefore gives the completed Gysin sequence

$$\begin{aligned} \cdots \to H^{q-2c}_{et}(Z,\Lambda(-c)) \to H^q_{et}(X,\Lambda) \to H^q_{et}(U,\Lambda) \\ \to H^{q-2c+1}_{et}(Z,\Lambda(-c)) \to \cdots . \end{aligned}$$

For the coefficients Λ, homotopy invariance gives $H^0_{et}(\mathbb{A}^d,\Lambda) = \Lambda$ and $H^q_{et}(\mathbb{A}^d,\Lambda) = 0$ for $q>0$. Apply the Gysin sequence to $\mathbb{P}^{d-1} \subset \mathbb{P}^d$. The map on degree-zero cohomology to $\mathbb{A}^d$ is an isomorphism; in positive degrees the sequence identifies cohomology with that of $\mathbb{P}^{d-1}$, shifted by two and twisted by -1. Induction, starting with a point, gives

$$H^q_{et}(\mathbb{P}^d,\Lambda) = \begin{cases} \Lambda(-j) & q=2j, \quad 0 \le j \le d, \\ 0 & \text{otherwise.} \end{cases}$$

The successive generators are powers of the hyperplane class. Purity and homotopy invariance are quoted standard etale-cohomology inputs [Mil]; the induction is the missing calculation in the draft.

C.3 Artin–Schreier in characteristic p

Suppose $\operatorname{char} k = p$. For any k-scheme, the Artin–Schreier sequence is

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \mathcal{O} \xrightarrow{F-1} \mathcal{O} \longrightarrow 0$$

on the etale site. Quasi-coherent cohomology agrees with Zariski cohomology. Affine acyclicity consequently gives

$$H^0_{et}(\mathbb{A}^1, \mathbb{F}_p) = \mathbb{F}_p, \qquad H^1_{et}(\mathbb{A}^1, \mathbb{F}_p) = k[t]/\{f^p - f \mid f \in k[t]\},$$

as additive groups, and $H^q_{et}(\mathbb{A}^1, \mathbb{F}_p) = 0$ for $q \geq 2$. Every class has a unique representative

$$\sum_{\substack{m>0 \\ p\nmid m}} a_m t^m$$

with finite support. To see existence, discard the constant using surjectivity of $a \mapsto a^p - a$ on k, and replace at^{pm} by $a^{1/p}t^m$ modulo $F - 1$, repeatedly. For uniqueness, a nonconstant polynomial $f^p - f$ has highest degree divisible by p; a nonzero polynomial in the displayed form has highest degree prime to p. Thus the quotient is, additively, $\bigoplus_{m>0,\, p\nmid m} k$. This identification is $\mathbb{F}_p$-linear; the image of $F - 1$ is not a k-subspace.

On $\mathbb{P}^d$, one has $H^0(\mathcal{O}) = k$ and $H^q(\mathcal{O}) = 0$ for all $q > 0$. Artin–Schreier gives

$$H^0_{et}(\mathbb{P}^d, \mathbb{F}_p) = \mathbb{F}_p, \qquad H^q_{et}(\mathbb{P}^d, \mathbb{F}_p) = 0 \quad (q > 0).$$

This includes both missing degree-one affine-line and projective-line calculations and explains why the prime-to-characteristic formulas cannot be used with $n = p$.

Appendix D

Integral transforms and linear categories

All fiber products and tensor products in this appendix are derived. We work with stable presentable infinity-categories and colimit-preserving linear functors. The kernel equivalence in this setting concerns enhanced functors; it does not prove representability for an arbitrary exact functor of triangulated categories.

D.1 The affine calculation

Let R be a commutative derived ring, and let A, B be commutative R-algebras. Write Mod_A for its derived category of modules, retaining its enhancement.

Proposition D.1.1. *Evaluation at A gives an equivalence*

$$\mathrm{Fun}^L_R(\mathrm{Mod}_A, \mathrm{Mod}_B) \simeq \mathrm{Mod}_{A^{op} \otimes_R B}\,.$$

For commutative A, the right side is $\mathfrak{QCoh}(\mathrm{Spec}\, A \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, B)$.

Proof. For F, the object $P = F(A)$ has its B-module structure and the commuting right A-action induced by endomorphisms of the free left A-module A. An R-linear colimit-preserving functor commutes with tensoring by R-modules, so for a free module $A \otimes_R V$ there is a natural equivalence

$$F(A \otimes_R V) \simeq P \otimes_R V.$$

Resolve any M by its simplicial free-module bar resolution. In degree n its underlying term is $A \otimes_R A^{\otimes_R n} \otimes_R M$. The faces are multiplication in A and

the action on M, and the degeneracies insert the unit. The right A-action on P identifies the image under F of every face and degeneracy with those of the two-sided bar construction for $P \otimes_A M$. Taking geometric realization gives a natural equivalence $F(M) \simeq P \otimes_A M$.

A natural transformation between two such functors restricts at A to a bimodule morphism. Conversely a bimodule morphism induces a map of the whole bar resolutions and hence a natural transformation. This identifies mapping objects, not just isomorphism classes of functors, and proves the equivalence. □

The tensor-product version is

$$\mathrm{Mod}_A \otimes_{\mathrm{Mod}_R} \mathrm{Mod}_B \simeq \mathrm{Mod}_{A \otimes_R B}\,.$$

Indeed an R-balanced functor preserving colimits in each variable is determined by its value at (A, B), with the two commuting actions. The same bar-resolution argument extends that datum uniquely to all modules. This is the universal property of the displayed equivalence. These calculations are the affine derived Morita argument [Toë07].

D.2 Perfect stacks and external products

We use perfect stacks in the sense of [BFN10]: they have affine diagonal and their quasi-coherent category is generated under colimits by perfect complexes, which are exactly its compact objects. Quasi-compact separated derived schemes are examples. Categories linear over X are module categories over $\mathfrak{QCoh}(X)$ in the symmetric monoidal category of presentable stable categories. The tensor product of small stable categories below includes idempotent completion.

For perfect complexes M_i, N_i on X_i, write RHom for the mapping complex over the base ring, not for the mapping space. The external-product calculation is

$$\begin{aligned}
&\mathrm{RHom}_{X_1 \times X_2}(M_1 \boxtimes M_2, N_1 \boxtimes N_2) \\
&\quad \simeq R\Gamma\big(X_1 \times X_2, p_1^*(M_1^\vee \otimes N_1) \otimes p_2^*(M_2^\vee \otimes N_2)\big) \\
&\quad \simeq \mathrm{RHom}_{X_1}(M_1, N_1) \otimes_R \mathrm{RHom}_{X_2}(M_2, N_2).
\end{aligned}$$

For quasi-compact separated schemes, a finite affine cover computes quasi-coherent cohomology by its Cech complex. Applying the affine calculation on each pair of intersections proves this derived Kunneth identity; finite totalizations

commute with tensor in a stable category. It proves full faithfulness on external products. Essential surjectivity also needs generation of the product category by these objects. With this compact-generation input, finite cones and retracts give

$$\mathrm{Perf}(X_1) \otimes_R \mathrm{Perf}(X_2) \simeq \mathrm{Perf}(X_1 \times_R X_2).$$

Applying the symmetric monoidal Ind functor gives the corresponding identity for quasi-coherent categories. The perfect-stack extension, including its generation input, is the theorem cited next.

Theorem D.2.1. *For maps of perfect stacks $X \to S \leftarrow Y$,*

$$\mathfrak{QCoh}(X \times_S Y) \simeq \mathfrak{QCoh}(X) \otimes_{\mathfrak{QCoh}(S)} \mathfrak{QCoh}(Y).$$

If $X \to S$ is a perfect morphism and S has affine diagonal, then for arbitrary $Y \to S$,

$$\mathfrak{QCoh}(X \times_S Y) \simeq \mathrm{Fun}^L_{\mathfrak{QCoh}(S)}(\mathfrak{QCoh}(X), \mathfrak{QCoh}(Y)).$$

These are quoted from [BFN10, Theorem 1.2]. The functor corresponding to a kernel P is $M \mapsto p_{Y*}(p_X^* M \otimes P)$. Relative self-duality of $\mathfrak{QCoh}(X)$ over $\mathfrak{QCoh}(S)$, under these perfect-stack hypotheses, turns the tensor statement into the functor statement.

D.3 The bar and monad comparison

The author's proposed global proof starts from

$$X \times_S Y \longrightarrow \big(X \times Y \rightrightarrows X \times S \times Y \longrightarrow \cdots\big).$$

After applying pullback on quasi-coherent sheaves and the external-product identifications, this is the augmented two-sided bar construction with terms

$$\mathfrak{QCoh}(X) \otimes \mathfrak{QCoh}(S)^{\otimes n} \otimes \mathfrak{QCoh}(Y).$$

Its geometric realization defines a comparison functor

$$\mathfrak{QCoh}(X) \otimes_{\mathfrak{QCoh}(S)} \mathfrak{QCoh}(Y) \longrightarrow \mathfrak{QCoh}(X \times_S Y).$$

Passing to right adjoints turns the colimit in Pr^L into a limit in Pr^R; the transition functors must also be replaced by their right adjoints. Denote by τ^* the map

from the degree-zero category to the realization, and by τ_* its right adjoint. If π is the map to $X \times Y$, the candidate monads on $\mathfrak{QCoh}(X \times Y)$ are

$$T_1 = \tau_* \tau^*, \qquad T_2 = \pi_* \pi^*.$$

The comparison must preserve their units and multiplications. For affine schemes, the monads are tensoring with the same derived algebra over the product ring. The bar faces identify their multiplication and the degeneracies identify their units. Restriction of scalars is conservative and preserves geometric realizations, so the monadic criterion applies and recovers the affine equivalence proved above.

For general stacks, one must additionally establish conservativity and the required realization preservation for the right adjoints, together with the compatible base-change maps for all simplicial terms. Objectwise base change by itself proves none of these assertions. The perfect-stack theorem supplies the result, but these lines are not an independent proof of its global descent step. This states exactly what remains of the original "Details needed" calculation.

D.4 Beyond perfect stacks

The original suggestion to remove perfectness by a completed tensor product remains an extension problem. One must specify the complete accessible t-structures, the completion functor, continuity conditions and a compatible descent theorem. It is not a consequence of the ordinary tensor-product identity. This direction is also discussed in [BFN10, Remark 1.3]. No theorem about arbitrary imperfect stacks is used in the notes.

Bibliography

[AKW14] Benjamin Antieau, Daniel Krashen, and Matthew Ward. *Derived categories of torsors for abelian schemes*. 2014. arXiv: 1409.2580 [math.AG].

[AT14] Nicolas Addington and Richard Thomas. "Hodge theory and derived categories of cubic fourfolds". In: *Duke Mathematical Journal* 163.10 (July 2014). ISSN: 0012-7094. DOI: 10.1215/00127094-2738639. URL: http://dx.doi.org/10.1215/00127094-2738639.

[Bay+21] Arend Bayer et al. "Stability conditions in families". In: *Publications mathématiques de l'IHÉS* 133.1 (May 2021), pp. 157–325. ISSN: 1618-1913. DOI: 10.1007/s10240-021-00124-6. URL: http://dx.doi.org/10.1007/s10240-021-00124-6.

[BAY+23] Arend BAYER et al. "Stability conditions on Kuznetsov components". In: *Annales scientifiques de l'École Normale Supérieure* 56.2 (2023), pp. 517–570. ISSN: 1873-2151. DOI: 10.24033/asens.2539. URL: http://dx.doi.org/10.24033/asens.2539.

[BB02] Alexei Bondal and Michel Van den Bergh. *Generators and representability of functors in commutative and noncommutative geometry*. 2002. arXiv: math/0204218 [math.AG].

[BFN10] David Ben-Zvi, John Francis, and David Nadler. "Integral transforms and Drinfeld centers in derived algebraic geometry". In: *Journal of the American Mathematical Society* 23.4 (2010), pp. 909–966. arXiv: 0805.0157.

[Bra21] Daniel Bragg. *Derived equivalences of twisted supersingular K3 surfaces*. 2021. arXiv: 1811.07379 [math.AG].

[Cha12] François Charles. *A remark on the Torelli theorem for cubic fourfolds*. 2012. arXiv: 1209.4509 [math.AG].

[DI81] Pierre Deligne and Luc Illusie. "Relevement des surfaces K3 en caractéristique nulle". In: *Surfaces Algébriques: Séminaire de Géométrie Algébrique d'Orsay 1976–78*. Springer. 1981, pp. 58–79.

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge University Press, 2010.

[Huy06] Daniel Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Clarendon Press, 2006.

[Huy16] Daniel Huybrechts. *Lectures on K3 surfaces*. Vol. 158. Cambridge University Press, 2016.

[KM07] A. Kuznetsov and D. Markushevich. *Symplectic structures on moduli spaces of sheaves via the Atiyah class*. 2007. arXiv: math/0703264 [math.AG].

[Kuz17] Alexander Kuznetsov. "Calabi–Yau and fractional Calabi–Yau categories". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2019.753 (Mar. 2017), pp. 239–267. ISSN: 0075-4102. DOI: 10.1515/crelle-2017-0004. URL: http://dx.doi.org/10.1515/crelle-2017-0004.

[LM19] Max Lieblich and Davesh Maulik. *A note on the cone conjecture for K3 surfaces in positive characteristic*. 2019. arXiv: 1102.3377 [math.AG].

[LO14] Max Lieblich and Martin Olsson. *Fourier-Mukai partners of K3 surfaces in positive characteristic*. 2014. arXiv: 1112.5114 [math.AG].

[LPZ18] Chunyi Li, Laura Pertusi, and Xiaolei Zhao. *Twisted cubics on cubic fourfolds and stability conditions*. 2018. arXiv: 1802.01134.

[Mil] James S. Milne. *Lectures on Etale Cohomology*. URL: https://www.jmilne.org/math/CourseNotes/LEC.pdf (visited on 09/08/2026).

[Ogu79] Arthur Ogus. "Supersingular K3 crystals". In: *Journées de Géométrie Algébrique de Rennes (Rennes, 1978)* 2 (1979), pp. 3–86.

[Orl03] Dmitri Orlov. "Derived categories of coherent sheaves and equivalences between them". In: *Russian Mathematical Surveys* 58.3 (2003), pp. 89–172.

[Toë07] Bertrand Toën. "The homotopy theory of dg-categories and derived Morita theory". In: *Inventiones mathematicae* 167.3 (2007), pp. 615–667.

[Yos01] Kōta Yoshioka. "Moduli spaces of stable sheaves on abelian surfaces". In: *Mathematische Annalen* 321 (2001), pp. 817–884.