Introduction to Hall algebras

Shuai Zong

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1 Classical Hall algebras

What algebraic structure can one extract from an abelian category $\mathcal{A}$? A familiar answer is the Grothendieck group $K_0(\mathcal{A})$: every short exact sequence $0 \to N \to E \to M \to 0$ gives the relation

$$[E]_{K_0} = [M]_{K_0} + [N]_{K_0}.$$

This records additivity across extensions but gives the same class to the middle terms of split and nonsplit extensions with the same end terms. A Hall algebra uses that extension data to construct a multiplication. Informally, one would like to write

$$[M][N] \sim \sum_{0 \to N \to E \to M \to 0} [E].$$

The multiplicity of $[E]$ must count its subobjects isomorphic to N with quotient M. This leads to a generally noncommutative product.

1.1 The Hall product

Definition 1.1. An essentially small abelian category $\mathcal{A}$ is *finitary* if $\mathrm{Hom}(M,N)$ and $\mathrm{Ext}^1(M,N)$ are finite for all $M,N\in\mathcal{A}$.

For a finitary category, let

$$\mathcal{H}(\mathcal{A})=\bigoplus_{[M]}\mathbb{Q}[M]$$

be the vector space whose direct sum ranges over the isomorphism classes of objects. Unlike $K_0(\mathcal{A})$, it keeps distinct isomorphism classes as separate basis vectors.

Definition 1.2. For objects M,N, define the *Hall product* by

$$F^E_{M,N}=\#\{N'\subset E\mid N'\simeq N,\ E/N'\simeq M\},\qquad [M][N]=\sum_{[E]}F^E_{M,N}[E].$$

Each coefficient is finite because subobjects of E isomorphic to N are represented by monomorphisms in the finite set $\mathrm{Hom}(N,E)$. Only finitely many middle terms occur, since each is the middle term of a class in $\mathrm{Ext}^1(M,N)$.

Remark 1.3. The multiplication above is the untwisted Hall product: the coefficient of $[E]$ is exactly $F^E_{M,N}$, with no Euler-form factor. We introduce the Euler twist only when comparing the A_2 calculation with the quantum Serre relations below.

Example 1.4 (Vector spaces over a finite field)**.** Let $\mathcal{A}$ be the category of finite-dimensional vector spaces over $\mathbb{F}_q$. Every short exact sequence

$$0\longrightarrow\mathbb{F}_q^n\longrightarrow E\longrightarrow\mathbb{F}_q^m\longrightarrow 0$$

splits, so $E\simeq\mathbb{F}_q^{m+n}$. The coefficient of $[\mathbb{F}_q^{m+n}]$ in $[\mathbb{F}_q^m][\mathbb{F}_q^n]$ counts n-dimensional subspaces of $\mathbb{F}_q^{m+n}$. There are $\prod_{i=0}^{n-1}(q^{m+n}-q^i)$ ordered linearly independent n-tuples in that space, and each n-dimensional subspace has $\prod_{i=0}^{n-1}(q^n-q^i)$ ordered bases. Dividing gives

$$F^{\mathbb{F}_q^{m+n}}_{\mathbb{F}_q^m,\mathbb{F}_q^n}=\#\,\mathrm{Gr}(n,m+n)(\mathbb{F}_q)=\frac{\prod_{i=0}^{n-1}(q^{m+n}-q^i)}{\prod_{i=0}^{n-1}(q^n-q^i)}=\begin{bmatrix}m+n\\ n\end{bmatrix}_q.$$

Thus the product is multiplication by a Gaussian binomial coefficient:

$$[\mathbb{F}_q^m][\mathbb{F}_q^n]=\begin{bmatrix}m+n\\ n\end{bmatrix}_q[\mathbb{F}_q^{m+n}].$$

In particular, $[\mathbb{F}_q]^2=(q+1)[\mathbb{F}_q^2]$.

An iterated Hall product counts flags of subobjects. The two ways of choosing a two-step flag give the associativity of the product.

Proposition 1.5. *The Hall product makes $\mathcal{H}(\mathcal{A})$ an associative unital $\mathbb{Q}$-algebra, with unit $[0]$.*

Proof. Fix $L,M,N,E\in\mathcal{A}$. The coefficient of $[E]$ in $([L][M])[N]$ first chooses $N'\subset E$, then a subobject of E/N' whose quotient is L. The inverse image of the latter subobject gives a flag

$$0\subset N'\subset M'\subset E,\qquad N'\simeq N,\quad M'/N'\simeq M,\quad E/M'\simeq L.$$

The coefficient in $[L]([M][N])$ chooses M' first and then $N' \subset M'$, so it counts the same flags. Thus the two products have the same coefficient at every $[E]$. The unique zero subobject and the unique subobject with zero quotient show that $[0][E] = [E] = [E][0]$. $\square$

1.2 Quiver representations

A standard source of finitary abelian categories is the representation theory of finite quivers over finite fields. The arrows allow nonsplit extensions, so their Hall products need not commute.

Definition 1.6. A *finite quiver* is a tuple $Q = (Q_0, Q_1, s, t)$ of finite sets of vertices Q_0 and arrows Q_1, together with source and target maps $s, t\colon Q_1 \to Q_0$. We draw an arrow a as $s(a) \xrightarrow{a} t(a)$.

Over a field k, a representation assigns a finite-dimensional k-vector space V_i to each vertex i and a linear map $V_a\colon V_{s(a)} \to V_{t(a)}$ to each arrow a. A morphism $V \to W$ is a family $f_i\colon V_i \to W_i$ satisfying $W_a f_{s(a)} = f_{t(a)} V_a$ for every arrow. These representations form the abelian category $\mathrm{Rep}_k(Q)$, with exact sequences computed vertex by vertex. When $k = \mathbb{F}_q$, this category is finitary.

Example 1.7 (The A_2 quiver)**.** The A_2 quiver has vertices $1, 2$ and a single arrow $1 \to 2$. A representation over $k = \mathbb{F}_q$ is a linear map $f\colon V_1 \to V_2$. Up to isomorphism, its indecomposable representations are the two simples S_1, S_2 and E:

$$S_1 = (k \to 0), \qquad S_2 = (0 \to k), \qquad E = (k \xrightarrow{\mathrm{id}} k).$$

Choosing complements to $\ker(f)$ and $\mathrm{im}(f)$ gives

$$(V_1 \xrightarrow{f} V_2) \simeq S_1^{\oplus \dim \ker(f)} \oplus E^{\oplus \mathrm{rank}(f)} \oplus S_2^{\oplus \dim \mathrm{coker}(f)}.$$

The sequence $0 \to S_2 \to E \to S_1 \to 0$ is nonsplit. Both E and $S_1 \oplus S_2$ contain a unique subrepresentation S_2 with quotient S_1. A subrepresentation S_1 with quotient S_2 exists only in $S_1 \oplus S_2$. Consequently,

$$[S_1][S_2] = [E] + [S_1 \oplus S_2], \qquad [S_2][S_1] = [S_1 \oplus S_2],$$

so $[S_1][S_2] - [S_2][S_1] = [E]$.

Next consider $[S_1][E]$ and $[E][S_1]$. A middle term has dimension vector $(2, 1)$, and the presence of E as a subrepresentation or quotient forces its arrow $k^2 \to k$ to have rank one. By the decomposition above, it is therefore isomorphic to $S_1 \oplus E$. To count its E-subrepresentations, choose a line in k^2 on which the arrow is nonzero. Any line other than the kernel works, giving $(q + 1) - 1 = q$ choices. An S_1-subrepresentation, on the other hand, must be the kernel, so

$$[S_1][E] = q[S_1 \oplus E], \qquad [E][S_1] = [S_1 \oplus E].$$

For $[E][S_2]$ and $[S_2][E]$, the same rank argument leaves only the middle term $E \oplus S_2$, represented by an injective arrow $k \to k^2$. A subrepresentation S_2 has quotient E exactly when

its line differs from the image of this arrow; there are q such lines. An E-subrepresentation must use the image itself. Thus

$$[E][S_2] = q[S_2 \oplus E], \qquad [S_2][E] = [S_2 \oplus E].$$

With $x = [S_1]$, $y = [S_2]$, and $z = [E]$, these products read

$$xy - yx = z, \qquad xz = qzx, \qquad zy = qyz.$$

Proposition 1.8. *For $x = [S_1]$ and $y = [S_2]$ in the Hall algebra of the A_2 quiver,*

$$x^2y - (q+1)xyx + qyx^2 = 0, \qquad xy^2 - (q+1)yxy + qy^2x = 0.$$

Proof. The preceding products give $xy = yx + z$, $xz = qzx$, and $zy = qyz$. For the first relation,

$$x^2y = xyx + qzx, \qquad xyx = yx^2 + zx.$$

Similarly, for the second,

$$xy^2 = yxy + qyz, \qquad yxy = y^2x + yz.$$

Substitution in each pair proves the two identities. □

These relations resemble the quantum Serre relations, but their coefficients differ. To obtain the quantum-group normalization, let $v^2 = q$ and twist the product by the Euler form of the acyclic quiver:

$$\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{a \in Q_1} d_{s(a)} e_{t(a)}, \qquad [M] \star [N] = v^{\langle \dim M, \dim N \rangle} [M][N], \quad \text{over } \mathbb{Q}(v).$$

Dimension vectors add in exact sequences, so bilinearity of the Euler form makes $\star$ associative. In type A_2, $\langle \dim S_1, \dim S_2 \rangle = -1$ and $\langle \dim S_2, \dim S_1 \rangle = 0$. Together with the diagonal pairings $\langle \dim S_i, \dim S_i \rangle = 1$, this gives $x \star y = v^{-1}xy$, $y \star x = yx$, and

$$\begin{aligned} &x \star x \star y = v^{-1}x^2y, \quad x \star y \star x = xyx, \quad y \star x \star x = vyx^2, \\ &x \star y \star y = v^{-1}xy^2, \quad y \star x \star y = yxy, \quad y \star y \star x = vy^2x. \end{aligned}$$

Multiplying the two untwisted relations by v^{-1} and using $v^{-1}(q+1) = v + v^{-1}$, we obtain

$$x \star x \star y - (v + v^{-1}) x \star y \star x + y \star x \star x = 0, \qquad x \star y \star y - (v + v^{-1}) y \star x \star y + y \star y \star x = 0.$$

These are the quantum Serre relations for $U_v^+(\mathfrak{sl}_3)$.

For a Dynkin quiver, the Hall numbers are polynomials in q. Write $F^E_{M,N}(T)$ for the polynomial whose value at q is the corresponding Hall number. Taking v to be an indeterminate gives the generic twisted Hall algebra $\mathcal{H}_v(Q)$ over $\mathbb{Q}(v)$, with multiplication

$$[M] \star [N] = v^{\langle \dim M, \dim N \rangle} \sum_{[E]} F^E_{M,N}(v^2)[E].$$

Let $\mathfrak{g}_Q$ be the semisimple Lie algebra associated with the underlying Dynkin diagram, and let e_i be the positive Chevalley generator at vertex i. Ringel's theorem identifies this generic algebra [Rin90].

Theorem 1.9 (Ringel)**.** *For a Dynkin quiver Q, the assignment*

$$U_v^+(\mathfrak{g}_Q) \longrightarrow \mathcal{H}_v(Q), \qquad e_i \longmapsto [S_i],$$

extends to an isomorphism of $\mathbb{Q}(v)$-algebras. A representation of dimension vector (d_i) corresponds to weight $\sum_i d_i\alpha_i$, where α_i is the simple root at vertex i.

Thus the simple classes generate the entire Hall algebra, with the quantum Serre relations as defining relations; for vertices not joined by an edge, these are commutation relations. For A_2, the two cubic relations above give a presentation of $\mathcal{H}_v(A_2)$ on x, y; every representation class is a polynomial in them, including

$$[E] = v\,x \star y - y \star x.$$

Conversely, the isomorphism gives the quantum group a basis indexed by isomorphism classes of quiver representations. The Hall multiplication depends on the orientation of Q, but its resulting algebra is isomorphic to the same $U_v^+(\mathfrak{g}_Q)$ for every orientation of the Dynkin diagram.

Example 1.10 (The nilpotent Jordan quiver)**.** The Jordan quiver has one vertex and one loop. A representation is a finite-dimensional vector space V with an endomorphism $t\colon V \to V$, equivalently a finite-dimensional $k[t]$-module. The structure theorem for modules over the principal ideal domain $k[t]$ decomposes a finite-dimensional representation into primary summands for irreducible polynomials in $k[t]$. We focus on the t-primary part, equivalently on nilpotent endomorphisms. Its indecomposables are $J_r = k[t]/(t^r)$ for $r \geq 1$. The isomorphism classes of nilpotent representations are thus indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, with

$$J_\lambda = \bigoplus_{i=1}^{\ell} J_{\lambda_i}.$$

Since $\dim J_\lambda = |\lambda| = \sum_i \lambda_i$, only partitions of $|\mu| + |\nu|$ can occur in the product $[J_\mu][J_\nu]$.

Proposition 1.11. *Over $k = \mathbb{F}_q$, the first two products in the nilpotent Jordan Hall algebra are*

$$[J_1]^2 = [J_2] + (q+1)[J_1 \oplus J_1], \qquad [J_1][J_2] = [J_3] + q[J_2 \oplus J_1].$$

Proof. The module J_2 has one submodule of type J_1, while all $q+1$ lines in $J_1 \oplus J_1$ are submodules. This gives the first formula. For the second product, the partitions of 3 give the possible middle terms J_3, $J_2 \oplus J_1$, and $J_1^{\oplus 3}$. The last cannot contain J_2, since t acts as zero, while J_3 contains the unique required submodule tJ_3. It remains to count the J_2-submodules of $J_2 \oplus J_1$. Choose a basis e_1, e_2, f with $te_1 = e_2$ and $te_2 = tf = 0$. A generator $u = ae_1 + be_2 + cf$ must satisfy $tu \neq 0$, hence $a \neq 0$, and

$$\langle u, tu\rangle = \langle e_1 + (c/a)f, e_2\rangle.$$

Thus these submodules are precisely the planes $\langle e_1 + cf, e_2\rangle$ for $c \in \mathbb{F}_q$. Their images modulo $\langle e_2\rangle$ are distinct lines, so there are exactly q of them. □

Here the Euler form vanishes, so the twisted and untwisted products agree. Indeed, the resolution $0 \to k[t] \xrightarrow{t^r} k[t] \to J_r \to 0$ gives

$$\dim_k \operatorname{Hom}(J_r, J_s) = \dim_k \operatorname{Ext}^1(J_r, J_s) = \min(r, s),$$

and additivity gives $\langle m, n\rangle = mn - mn = 0$ for arbitrary nilpotent representations of dimensions m, n.

As in the Dynkin case, the coefficients $F^{J_\lambda}_{J_\mu,J_\nu}$ are polynomials in q. Writing these polynomials as $F^{\lambda}_{\mu,\nu}(T)$, we obtain the generic algebra $\mathcal{H}^{\mathrm{nil}}_T$ over $\mathbb{Q}(T)$, with basis u_λ indexed by partitions and multiplication

$$u_\mu u_\nu = \sum_\lambda F^{\lambda}_{\mu,\nu}(T) u_\lambda, \qquad F^{\lambda}_{\mu,\nu}(q) = F^{J_\lambda}_{J_\mu,J_\nu}.$$

Let $\Lambda_{\mathbb{Q}(T)} = \mathbb{Q}(T)[e_1, e_2, \ldots]$, where e_r is the elementary symmetric function of degree r. The following theorem identifies the Hall product with multiplication of symmetric functions [Mac95, Chapters II–III]; see also [Sch06, Theorem 2.6 and Section 2.4].

Theorem 1.12 (Steinitz–Hall)**.** *The algebra $\mathcal{H}^{\mathrm{nil}}_T$ is commutative and freely generated by the semisimple classes $u_{(1^r)}$, for $r \geq 1$. There is a graded $\mathbb{Q}(T)$-algebra isomorphism*

$$\Phi\colon \mathcal{H}^{\mathrm{nil}}_T \xrightarrow{\sim} \Lambda_{\mathbb{Q}(T)}, \qquad u_{(1^r)} \longmapsto T^{-r(r-1)/2} e_r.$$

The grading on the Hall side is total dimension: $\deg u_\lambda = |\lambda|$. The theorem distinguishes two roles for partitions: all u_λ form a vector-space basis, while only the column partitions (1^r) are needed as algebra generators. For example, the first product computed above reads

$$u_{(1)}^2 = u_{(2)} + (T+1) u_{(1,1)},$$

so its image under Φ gives

$$\Phi(u_{(2)}) = e_1^2 - (1 + T^{-1}) e_2.$$

The images of the general partition classes are scalar multiples of the Hall–Littlewood symmetric functions at parameter T^{-1}; the displayed normalization of the generators fixes this identification. Hall polynomials therefore describe the multiplication coefficients of this symmetric-function basis.

2 Hall algebras from spans

The A_2 and Jordan examples relate Hall algebras to quantum groups and symmetric functions by counting extensions over a finite field. To construct geometric versions, such as cohomological Hall algebras, we need a description in which extension spaces can be used even when their sets of points are infinite. We first rewrite the classical multiplication as convolution, separate its extension correspondence from the counting operation, and then formulate these two steps using spans and a linearization.

2.1 Convolution and spans

For this convolution description, $\mathcal{A}$ is still an essentially small finitary abelian category, and we use the untwisted Hall algebra over $\mathbb{Q}$ from Section 1. Let $\mathcal{A}^{\simeq}$ be its groupoid of objects

and isomorphisms, and write Fun_c for functions with finite support on isomorphism classes. There is a natural vector-space identification

$$\mathcal{H}(\mathcal{A}) = \bigoplus_{[M]} \mathbb{Q}[M] \xrightarrow{\sim} \mathrm{Fun}_c(\pi_0 \mathcal{A}^{\simeq}, \mathbb{Q}), \qquad [M] \longmapsto \delta_M.$$

Under this identification, the Hall multiplication is convolution:

$$(f * g)(E) = \sum_{N' \subset E} f(E/N') g(N').$$

To express this sum geometrically, let $\mathcal{E}$ be the groupoid of short exact sequences in $\mathcal{A}$ and their isomorphisms. Recording the end terms and the middle term gives the correspondence

$$\mathcal{A}^{\simeq} \times \mathcal{A}^{\simeq} \xleftarrow{p} \mathcal{E} \xrightarrow{q} \mathcal{A}^{\simeq},$$

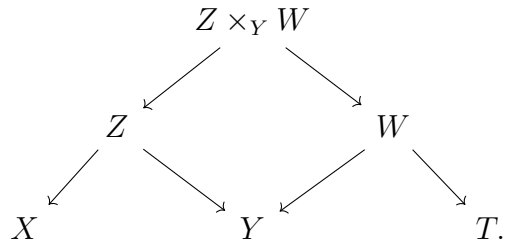
where $p(0 \to N \to E \to M \to 0) = (M, N)$ and $q(0 \to N \to E \to M \to 0) = E$. Pullback p^* evaluates the external product $(f \boxtimes g)(M, N) = f(M)g(N)$ on the end terms, and pushforward $q_!$ sums over the subobjects of the fixed middle term. Thus

$$f * g = q_! p^* (f \boxtimes g), \qquad q_! p^* (\delta_M \boxtimes \delta_N)(E) = F^E_{M,N}.$$

Finitarity makes these sums finite for finitely supported inputs and gives a finitely supported output. We will spell out the fibre calculation in the counting example below.

The Hall product thus separates into an extension correspondence and the operation of counting its fibres. To replace counting by a geometric invariant, we first regard the correspondence as a morphism in a category of spans; a suitable functor out of this category will then turn it into a product on that invariant.

Definition 2.1. Let $\mathcal{C}$ be an ∞-category with finite limits. The $(\infty, 2)$-category $\mathrm{Span}_2(\mathcal{C})$ has the objects of $\mathcal{C}$, spans $X \leftarrow Z \rightarrow Y$ as 1-morphisms, and maps between their middle objects over $X \times Y$ as 2-morphisms. Composition is given by pullback:



The outer legs form the composite span, and the identity is $X \xleftarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X$. Taking the maximal subgroupoid of each mapping ∞-category gives the ∞-category $\mathrm{Span}(\mathcal{C})$. Cartesian product in $\mathcal{C}$, applied to both objects and spans, makes it symmetric monoidal, with the terminal object as unit.

See [GR17, Chapter 7, Sections 1.1–1.2] for this construction. The Hall correspondence is now a morphism $\mathcal{A}^{\simeq} \otimes \mathcal{A}^{\simeq} \to \mathcal{A}^{\simeq}$ in spans. We want it to be the multiplication of an $\mathbb{E}_1$-algebra, so we must supply a unit and associativity compatible with every iterated product. The following theorem identifies exactly the simplicial data that do this; we define the 2-Segal condition next and explain the two constructions on objects.

Theorem 2.2 (2-Segal objects and algebras in spans)**.** *Let $\mathcal{C}$ have finite limits. A 2-Segal object $X_\bullet$ determines an $\mathbb{E}_1$-algebra on X_1 in $\mathrm{Span}(\mathcal{C})$, with multiplication and unit*

$$X_1 \times X_1 \xleftarrow{(d_0,d_2)} X_2 \xrightarrow{d_1} X_1, \qquad * \longleftarrow X_0 \xrightarrow{s_0} X_1.$$

Conversely, every $\mathbb{E}_1$-algebra in spans arises this way, and the two constructions are inverse up to equivalence [God25, Corollary 5.3].

2.2 The 2-Segal condition

We use the face-map form of the 2-Segal condition [Fel+21, Section 1].

Definition 2.3. Let $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object with face maps $d_i \colon X_n \to X_{n-1}$. It is *2-Segal* if, for every $n \geq 2$ and $0 < i < n$, both squares

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_{i+1}} & X_n \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ X_n & \xrightarrow[d_i]{} & X_{n-1} \end{array} \qquad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ {\scriptstyle d_{n+1}}\downarrow & & \downarrow{\scriptstyle d_n} \\ X_n & \xrightarrow[d_i]{} & X_{n-1} \end{array}$$

are Cartesian.

For any simplicial object, its face maps give a candidate multiplication span

$$\mu \colon X_1 \times X_1 \xleftarrow{(d_0,d_2)} X_2 \xrightarrow{d_1} X_1.$$

For three factors, the two parenthesizations have middle objects

$$P_{\mathrm{L}} = X_2 \underset{d_1, X_1, d_0}{\times} X_2, \qquad P_{\mathrm{R}} = X_2 \underset{d_1, X_1, d_2}{\times} X_2.$$

Each composite is a span from X_1^3 to X_1; its input maps record the three short edges, and its output map records the long edge. Write $\rho_{ij} \colon X_3 \to X_1$ for restriction to the edge (i, j). The two squares in Definition 2.3, with $n = 2$ and $i = 1$, give equivalences

$$P_{\mathrm{L}} \xleftarrow[\sim]{(d_0,d_2)} X_3 \xrightarrow[\sim]{(d_3,d_1)} P_{\mathrm{R}}.$$

Both composite spans are therefore identified with

$$X_1^3 \xleftarrow{(\rho_{23},\rho_{12},\rho_{01})} X_3 \xrightarrow{\rho_{03}} X_1.$$

This identifies the two parenthesizations as morphisms in spans and gives the first associativity comparison. The unit is represented by $* \longleftarrow X_0 \xrightarrow{s_0} X_1$ and follows from the 2-Segal condition [Fel+21, Theorem].

To obtain the $\mathbb{E}_1$-structure, we must make this calculation compatible with substitution of arbitrary products. Define the N-ary multiplication using X_N as middle object: its inputs are the consecutive edges in the order $(N-1, N), \ldots, (0, 1)$, and its output is the long edge $(0, N)$. For $N = n_1 + \cdots + n_k$, first multiplying within each block and then multiplying the k outputs gives middle object

$$X_k \times_{X_1^k} (X_{n_k} \times \cdots \times X_{n_1}).$$

Thus associativity reduces to identifying this pullback with X_N, compatibly with the input and output maps. The required identification is the following consequence of the two face-map squares.

Lemma 2.4. *Let $0 = b_0 < b_1 < \cdots < b_k = N$ and $n_j = b_j - b_{j-1}$. Restriction to the block endpoints and to each block induces an equivalence*

$$X_N \simeq X_k \times_{X_1^k} (X_{n_k} \times \cdots \times X_{n_1}),$$

where both maps to X_1^k record the long edges of the blocks in reverse order.

Proof. Write $A = X_1$. For $a_0 < \cdots < a_r$ in $[m]$, let $\rho^m_{a_0,\dots,a_r} \colon X_m \to X_r$ be restriction to those vertices. We first show that inserting one vertex into the i-th edge, for $1 \le i \le m$, gives a Cartesian square

$$\begin{array}{ccc} X_{m+1} & \xrightarrow{\rho^{m+1}_{i-1,i,i+1}} & X_2 \\ {\scriptstyle d_i}\downarrow & & \downarrow{\scriptstyle d_1} \\ X_m & \xrightarrow[\rho^m_{i-1,i}]{} & A. \end{array}$$

The square commutes because both composites restrict to the edge $(i-1, i+1)$ of $[m+1]$. To see that it is Cartesian, apply the second square of Definition 2.3 with $n = t$ and the same index i, for $t = m, m-1, \dots, i+1$. Pasting these squares removes the vertices to the right of $i+1$ and leaves the row $X_{i+1} \xrightarrow{d_i} X_i$. Next apply the first square with $n = t$ and index $t-1$, for $t = i, i-1, \dots, 2$. Pasting removes the vertices to the left of $i-1$ and leaves $X_2 \xrightarrow{d_1} X_1$. Transposing the resulting outer rectangle gives the displayed square; either sequence is empty when i is at the corresponding endpoint.

More generally, replace the i-th edge of $[m]$ by r edges. Set $M = m + r - 1$, and use the injections

$$\beta \colon [m] \hookrightarrow [M], \qquad \beta(j) = \begin{cases} j, & j < i, \\ j + r - 1, & j \ge i, \end{cases} \qquad \sigma \colon [r] \hookrightarrow [M], \quad \sigma(a) = i - 1 + a.$$

They give a commutative square

$$\begin{array}{ccc} X_M & \xrightarrow{\sigma^*} & X_r \\ {\scriptstyle \beta^*}\downarrow & & \downarrow{\scriptstyle \rho^r_{0,r}} \\ X_m & \xrightarrow[\rho^m_{i-1,i}]{} & A. \end{array}$$

It is Cartesian by induction on r. The case $r = 1$ is the identity square, and $r = 2$ is the preceding square. For $r > 2$, first expand to length $r-1$, then insert one vertex into its last edge, at position $i + r - 2$. The one-vertex square identifies this second step with a pullback by X_2; the same square applied to the last edge of X_{r-1} identifies X_r with $X_{r-1} \times_A X_2$. Pasting with the induction square gives $X_M \simeq X_m \times_A X_r$, with the maps shown above.

Now start with X_k and expand its edges from left to right. At step j, the current object is $X_{b_{j-1}+k-j+1}$; its $(b_{j-1}+1)$-st edge is the j-th original edge of $[k]$. Expanding it to length n_j gives, by the preceding Cartesian square,

$$X_{b_j+k-j} \simeq X_k \times_{A^j} (X_{n_j} \times \cdots \times X_{n_1}).$$

The maps to A^j are the first j edges of X_k and the long edge of each expanded block, both in reverse order. For $j = k$ we have $b_k = N$, so this is the stated equivalence. At every step the comparison maps are restrictions along the displayed injections; their composites are restriction to $\{b_0, \dots, b_k\}$ and to each consecutive block, as required. □

The lemma identifies the middle objects in the substitution calculation, and restriction to the consecutive and long edges identifies both legs. More generally, an iterated multiplication described by a planar rooted tree has an iterated pullback as its middle object. Applying the lemma at each vertex identifies that pullback with X_N. If one vertex of the tree is further subdivided, restricting directly from X_N or first to the larger block and then to its subblocks gives the same maps by simplicial functoriality. The universal property of the iterated pullbacks makes these comparisons compatible with tree refinements, including their higher homotopies. They provide the coherent associativity of the $\mathbb{E}_1$-multiplication; the unit above is supplied by the degeneracy map.

We now construct the inverse on objects. Starting from an $\mathbb{E}_1$-algebra A, we must recover a simplicial object, then check the two Cartesian face squares. Its terms are the middle objects of the coherent r-ary multiplication spans:

$$\mu_r \colon A^r \xleftarrow{\lambda_r} X_r \xrightarrow{\tau_r} A \qquad (r \geq 0).$$

Here $X_1 \simeq A$, and X_0 is the middle object of the unit span. The input edges are ordered from right to left, as in the forward construction. Write $\lambda_{m,p} \colon X_m \to A$ for the p-th input map, counting from the left in A^m. Substituting μ_r into that input of μ_m gives the basic comparison

$$X_{m+r-1} \simeq X_m \times_{\lambda_{m,p}, A, \tau_r} X_r \qquad (m, r \geq 1,\ 1 \leq p \leq m),$$

compatibly with the input and output maps.

The face maps come from merging two adjacent inputs or removing an end input. For $N \geq 2$ and $1 \leq j < N$, insert μ_2 into the $(N-j)$-th input of μ_{N-1}. The comparison above identifies X_N with the resulting pullback; its projection to X_{N-1} is the internal face d_j. For the two outer faces, use

$$\mu_N \simeq \mu_2(\mu_{N-1}, \mathrm{id}_A), \qquad \mu_N \simeq \mu_2(\mathrm{id}_A, \mu_{N-1})$$

and project to the middle object X_{N-1} of the displayed $(N-1)$-ary operation. These projections are d_0 and d_N, respectively. In degree one, the unit equivalences $\mu_2(\mu_0, \mathrm{id}_A) \simeq \mathrm{id}_A$ and $\mu_2(\mathrm{id}_A, \mu_0) \simeq \mathrm{id}_A$ identify X_1 with the corresponding pullbacks $X_2 \times_A X_0$. Their projections to X_0 give $d_0, d_1 \colon X_1 \to X_0$, respectively.

The degeneracies come from inserting a unit. For $0 \leq j \leq N$, insert μ_0 into the $(N+1-j)$-th input of μ_{N+1}. The unit comparison identifies X_N with $X_{N+1} \times_A X_0$, and projection to X_{N+1} gives s_j. The associativity and unit coherence of the $\mathbb{E}_1$-algebra give the simplicial identities for these face and degeneracy maps.

It remains to check the 2-Segal condition. For $n \geq 2$ and $0 < i < n$, inserting a binary operation at an internal edge and adjoining one at either end are compatible refinements of the same $(n-1)$-ary operation. Their common refinement has middle object X_{n+1}, giving equivalences

$$X_{n+1} \xrightarrow[\sim]{(d_{i+1}, d_0)} X_n \underset{d_0, X_{n-1}, d_i}{\times} X_n,$$

$$X_{n+1} \xrightarrow[\sim]{(d_i, d_{n+1})} X_n \underset{d_n, X_{n-1}, d_i}{\times} X_n$$

For example, the second fibre product combines the middle object of $\mu_2(\mathrm{id}_A, \mu_{n-1})$ with that of an internal binary substitution in μ_{n-1}, over their common X_{n-1}. Associativity identifies this iterated pullback with the middle object of μ_{n+1}. The first equivalence is the same calculation at the other end. Thus both face squares are Cartesian.

Starting from a 2-Segal object, this construction recovers its face maps from vertex restrictions and its degeneracies from the unit comparisons. Starting from A, it recovers the spans μ_r and their substitution equivalences. This completes the two constructions on objects in Theorem 2.2.

Remark 2.5. The full equivalence of ∞-categories requires a corresponding notion of morphism; the object-level argument above does not prove that extension. Gödicke uses *2-Segal spans* $X_\bullet \xleftarrow{s} M_\bullet \xrightarrow{t} Y_\bullet$, with Cartesian comparisons

$$M_n \simeq X_n \times_{X_1} M_1, \qquad M_n \simeq Y_n \times_{Y_1^n} M_1^n.$$

The first uses the long edge and the second uses the consecutive edges. In degree zero these mean $M_0 \simeq X_0 \times_{X_1} M_1$ and $M_0 \simeq Y_0$, with the first map to X_1 given by s_0. With these morphisms, the cited theorem gives [God25, Definition 2.2 and Corollary 5.3]

$$2\mathrm{Seg}^{\leftrightarrow}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Span}(\mathcal{C})).$$

2.3 The Waldhausen S-construction

The Waldhausen S-construction, also used in algebraic K-theory, organizes objects, exact sequences, and longer filtrations into a 2-Segal object. Its multiplication span is the extension correspondence above. We work with exact ∞-categories in the sense of [Bar15, Definition 3.1], which include ordinary exact categories and stable ∞-categories.

Construction 2.6. Let $\mathrm{Ar}[n]$ be the arrow category of $[n]$, with objects (i, j) for $0 \le i \le j \le n$. For an exact ∞-category $\mathcal{A}$, let $S_n(\mathcal{A})$ be the full subcategory of $\mathrm{Fun}(\mathrm{Ar}[n], \mathcal{A})$ consisting of diagrams F such that $F_{ii} \simeq 0$ and

$$\begin{array}{ccc} F_{ij} & \rightarrowtail & F_{ik} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & F_{jk} \end{array} \qquad (i \le j \le k)$$

is exact in the chosen exact structure. Restriction along maps in Δ, followed by taking the maximal ∞-groupoid in each degree, gives the simplicial space $S_\bullet(\mathcal{A})^{\simeq}$.

An object of $S_n(\mathcal{A})$ can be read as a filtration

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \cdots \hookrightarrow F_n$$

together with its coherent quotients $F_{ij} = F_j/F_i$. In low degrees, $S_0(\mathcal{A})^{\simeq} \simeq *$, $S_1(\mathcal{A})^{\simeq} \simeq \mathcal{A}^{\simeq}$, and $S_2(\mathcal{A})^{\simeq}$ parametrizes exact sequences. For an ordinary abelian category, the last is the groupoid $\mathcal{E}$ from the convolution construction, and the faces of $N \to E \to M$ are $d_0 = M$, $d_1 = E$, and $d_2 = N$.

In general, d_0 passes to the quotient filtration $0 \subset F_2/F_1 \subset \cdots \subset F_n/F_1$, each d_i with $0 < i < n$ omits F_i, and d_n omits the last stage. The quotient data and maps

on morphisms come from restriction of the exact diagrams. The degeneracy s_i repeats the stage F_i, inserting a zero successive quotient. In degree one, $s_0(E)$ is the sequence $0 \to 0 \to E \xrightarrow{\mathrm{id}} E \to 0$, while $s_1(E)$ is $0 \to E \xrightarrow{\mathrm{id}} E \to 0 \to 0$. Their end terms give the products $[E][0]$ and $[0][E]$, so the degeneracies recover the classical unit.

Dyckerhoff–Kapranov prove the following 2-Segal result [DK12, Theorem 7.3.3].

Theorem 2.7 (Dyckerhoff–Kapranov)**.** *The simplicial space $S_\bullet(\mathcal{A})^{\simeq}$ of an exact ∞-category is 2-Segal.*

Sketch of the two Cartesian squares. In the first square of Definition 2.3, the two restricted exact diagrams specify all stages except F_{i+1}, together with $F_{i+1}/F_1 \to F_{i+2}/F_1$. The missing stage is recovered by the homotopy pullback

$$F_{i+1} \simeq F_{i+2} \times_{F_{i+2}/F_1} (F_{i+1}/F_1).$$

The exact structure is stable under this pullback; its bicartesian squares recover the other required quotients. In the second square, d_{n+1} retains the flag through F_n, while d_i retains F_{n+1} and all earlier stages except F_i. Their common restriction is the flag through F_n with F_i omitted. The first diagram supplies F_i; the second supplies the final stage and the inclusion $F_n \to F_{n+1}$. Composing $F_i \to F_n \to F_{n+1}$ and completing the resulting exact squares recovers all quotients involving both stages. Both reconstructions are natural, with contractible spaces of compatible exact-square completions, so the restriction maps are equivalences of maximal ∞-groupoids. □

Thus the extension correspondence and the flags proving its associativity belong to a single 2-Segal object $S_\bullet(\mathcal{A})^{\simeq}$.

For cohomological Hall algebras, we need moduli of objects and extensions on which a cohomology or sheaf theory can act. The same S-construction produces these moduli by allowing the objects to vary in families. Let k be a field, and let CAlg_k be the category of connective derived commutative k-algebras. A derived prestack is a functor $\mathrm{CAlg}_k \to \mathcal{S}$, where $\mathcal{S}$ denotes spaces. Given a family of exact ∞-categories $R \mapsto \mathcal{A}_R$ with coherently compatible exact base-change functors, define

$$\mathfrak{S}_n \colon R \longmapsto S_n(\mathcal{A}_R)^{\simeq}.$$

Because limits of prestacks are computed pointwise, the preceding theorem makes $\mathfrak{S}_\bullet$ a 2-Segal prestack. For example, $\mathcal{A}_R = \mathrm{Perf}(R)$ gives perfect complexes and their cofiber sequences over $\operatorname{Spec} R$. More generally, for a fixed k-scheme X, take $\mathcal{A}_R = \mathrm{Perf}(X \times_k \operatorname{Spec} R)$. Then $\mathfrak{S}_1(R)$ parametrizes families of perfect complexes on X, and $\mathfrak{S}_2(R)$ parametrizes cofiber sequences of such families. The face maps again record the quotient, middle term, and subobject. Geometric Hall constructions for coherent sheaves use the corresponding derived moduli of objects and exact sequences [PS19, Section 4].

2.4 Three-functor formalisms

The span construction has separated extensions from the invariant we apply to them. To obtain a geometric Hall algebra, we now assign an invariant to each space, pull it back along the input map, and push it forward along the output map. The permitted output maps depend on the chosen invariant.

Definition 2.8. Let $\mathcal{E}$ be a class of maps in $\mathcal{C}$ containing equivalences and closed under composition, base change, and Cartesian products. Write $\mathrm{Span}(\mathcal{C},\mathcal{E})$ for spans whose right leg belongs to $\mathcal{E}$. A *three-functor formalism* is a lax symmetric monoidal functor [Sch26, Definition 2.4]

$$D\colon \mathrm{Span}(\mathcal{C},\mathcal{E}) \longrightarrow (\mathrm{Cat}_\infty,\times).$$

More generally, a lax symmetric monoidal functor to a symmetric monoidal ∞-category $\mathcal{V}$ will be called a *linearization*; taking $\mathcal{V}=\mathrm{Vect}_{\mathbb{Q}}$ allows numerical realizations as well.

The three operations are obtained from this functor as follows. For a map $f\colon X\to Y$, define pullback and, when $f\in\mathcal{E}$, pushforward by the elementary spans

$$Y \xleftarrow{f} X \xrightarrow{\mathrm{id}} X \longmapsto f^*\colon D(Y)\to D(X),$$
$$X \xleftarrow{\mathrm{id}} X \xrightarrow{f} Y \longmapsto f_!\colon D(X)\to D(Y).$$

Every span $X \xleftarrow{p} Z \xrightarrow{q} Y$ factors into these two types, so its image is $q_!p^*$. The lax monoidal structure supplies an external product $D(X)\otimes D(Y)\to D(X\times Y)$. Pullback along the diagonal gives the internal product

$$D(X)\otimes D(X) \longrightarrow D(X\times X) \xrightarrow{\Delta_X^*} D(X).$$

For category-valued D, the tensor symbol on the left denotes the Cartesian product of categories, and the resulting functor is the tensor product within $D(X)$. For functions it is pointwise multiplication.

A lax monoidal functor sends $\mathbb{E}_1$-algebras to $\mathbb{E}_1$-algebras. Consequently, when the multiplication and unit spans of a 2-Segal object are admissible, $D(X_1)$ inherits an $\mathbb{E}_1$-algebra structure with product

$$D(X_1)\otimes D(X_1) \longrightarrow D(X_1\times X_1) \xrightarrow{(d_0,d_2)^*} D(X_2) \xrightarrow{(d_1)_!} D(X_1).$$

This is the desired Hall product on the chosen invariant.

In practice, pullback and pushforward are often given before the functor on spans. For a numerical invariant, base change directly shows that pull–push respects composition of spans, as in the counting example below. For category-valued invariants, the comparison equivalences must also satisfy higher coherence. When pushforwards are right adjoints to pullbacks, the following universal property constructs this coherent extension from the canonical base-change maps.

Theorem 2.9 (The span universal property, right-adjoint case)**.** *Let $\mathcal{C}$ have finite limits and let $\mathcal{E}$ contain equivalences and be closed under composition and base change. Assume also that $\mathcal{E}$ is left cancellable: $gf,g\in\mathcal{E}$ implies $f\in\mathcal{E}$. Suppose $D^*\colon \mathcal{C}^{\mathrm{op}}\to\mathrm{Cat}_\infty$ has the following properties: each f^*, for $f\in\mathcal{E}$, has a right adjoint f_*, and for every Cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array} \qquad (f\in\mathcal{E}),$$

the canonical mate is an equivalence:

$$g^* f_* \longrightarrow f'_* g'^*.$$

Write $\mathrm{Span}_2(\mathcal{C}, \mathcal{E})$ *for the* $(\infty, 2)$*-category of spans with right leg in* $\mathcal{E}$*, retaining all maps between middle objects over the endpoints as 2-morphisms. Then* D^* *extends, with a contractible space of choices, to a 2-functor*

$$\widetilde{D}\colon \mathrm{Span}_2(\mathcal{C}, \mathcal{E})^{\mathrm{co}} \longrightarrow \mathbf{Cat}_\infty, \qquad (X \xleftarrow{p} Z \xrightarrow{q} Y) \longmapsto q_* p^*.$$

Here $\mathbf{Cat}_\infty$ *retains natural transformations as 2-morphisms, and* co *reverses 2-morphisms, not the direction of a span. More precisely, restriction induces an equivalence*

$$\mathbf{Fun}_2\Big(\mathrm{Span}_2(\mathcal{C}, \mathcal{E})^{\mathrm{co}}, \mathbf{Cat}_\infty\Big) \simeq \mathbf{Fun}_{\mathcal{E}\text{-radj}}\Big(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_\infty\Big).$$

Here the right-hand side consists of these right-adjointable diagrams and transformations whose squares along $\mathcal{E}$ *are right-adjointable [CLL26, Definition 2.4 and Theorem 4.20].*

Suppose further that each $D(X)$ *is symmetric monoidal, the pullbacks are symmetric monoidal coherently in* X*, and the canonical projection maps*

$$A \otimes f_* B \longrightarrow f_*(f^* A \otimes B) \qquad (A \in D(Y),\ B \in D(X))$$

are equivalences for $f\colon X \to Y$ *in* $\mathcal{E}$*. Then the extension is lax symmetric monoidal for Cartesian products of spans and categories [CLL26, Theorems 5.14 and 5.17].*

This is the case $I = \mathcal{C}^{\simeq}$, $P = E = \mathcal{E}$ of the cited results of Cnossen–Lenz–Linskens; in this case their iterated-span 2-category identifies with $\mathrm{Span}_2(\mathcal{C}, \mathcal{E})^{\mathrm{co}}$ above. Passing to the maximal subgroupoids of the mapping categories gives a functor $D\colon \mathrm{Span}(\mathcal{C}, \mathcal{E}) \to \mathrm{Cat}_\infty$. The canonical mate in the theorem is obtained from the commutative pullback square using the units and counits of $f^* \dashv f_*$ and $f'^* \dashv f'_*$. Thus its invertibility is a condition to check, while coherent extension to iterated spans is the conclusion of the theorem.

Example 2.10 (Quasi-coherent and derived Hall categories)**.** Take $D(X) = \mathrm{QCoh}(X)$ on quasi-compact quasi-separated derived schemes over a field k, with all morphisms admissible. Derived pullback Lf^* has right adjoint Rf_*, and derived Cartesian squares satisfy the base-change condition [Toë12, Proposition 1.4]. On affines, for $A \to B$, $A \to A'$, and $B' = B \otimes^{\mathbf{L}}_A A'$, this is the identity

$$A' \otimes^{\mathbf{L}}_A M \simeq B' \otimes^{\mathbf{L}}_B M \qquad (M \in \mathrm{Mod}_B),$$

viewed in $\mathrm{Mod}_{A'}$. The derived fibre product retains the Tor terms needed for this identity; no flatness assumption is imposed. Derived tensor products and symmetric monoidal pullbacks satisfy the projection formula, so the monoidal part of Theorem 2.9 supplies the lax symmetric monoidal span functor.

Hall moduli are stacks, so the next step is descent from schemes to geometric derived stacks and verification of the admissible legs. Porta–Sala carry this out for the derived stack $\mathfrak{M} = \mathrm{Coh}(S)$ of coherent sheaves on a smooth proper complex curve or surface S [PS19, Section 4.2]. If $\mathfrak{E}$ is its derived stack of extensions, the resulting product is

$$\begin{aligned} \mathrm{QCoh}(\mathfrak{M}) \times \mathrm{QCoh}(\mathfrak{M}) &\longrightarrow \mathrm{QCoh}(\mathfrak{M}), \\ (F, G) &\longmapsto Rq_* Lp^*(F \boxtimes G), \end{aligned}$$

for $\mathfrak{M}^2 \xleftarrow{p} \mathfrak{E} \xrightarrow{q} \mathfrak{M}$. The geometric S-construction and the lax monoidal realization make this an $\mathbb{E}_1$-monoidal ∞-category, including the associativity equivalences and all their higher compatibilities.

To restrict to bounded coherent complexes, one must also preserve boundedness and coherence. In this curve or surface setting, the input map is ind-derived-lci and the output map is representable by proper schemes; these give the required pullback and pushforward properties. Porta–Sala obtain the corresponding categorified Hall product on $\mathrm{Coh}^b_{\mathrm{pro}}(\mathfrak{M})$, where the pro-category accounts for the non-quasi-compact moduli stack [PS19, Theorem 4.9]. Thus the derived Hall category is constructed from an actual coherent span functor, rather than from an associativity equality in its Grothendieck group.

Example 2.11 (Counting by homotopy cardinality)**.** A space is *locally finite* if its homotopy groups at each point are finite and vanish in sufficiently high degrees; it is *finite* if it also has finitely many components. A map is finite if all its homotopy fibres are finite [GKT18b, Section 3.11]. On locally finite spaces, take

$$D(X) = \mathrm{Fun}(\pi_0 X, \mathbb{Q}),$$

the vector space of all rational-valued functions on components. Pullback is precomposition, and pushforward along a finite map $f\colon X \to Y$ is

$$(f_!\varphi)(y) = \sum_{[z]\in\pi_0(X_y)} \varphi(z) \prod_{i\geq 1} |\pi_i(X_y, z)|^{(-1)^i}.$$

The sum is finite by assumption. For an equivalence, the fibre is contractible, so $f_! = (f^*)^{-1}$. Cartesian base change identifies the fibres, giving $g^* f_! = f'_! g'^*$, and the fibre-cardinality formula gives $(gf)_! = g_! f_!$ [GKT18b, Lemma 3.5]. Thus $(X \xleftarrow{p} Z \xrightarrow{q} Y) \mapsto q_! p^*$ defines a functor on spans with finite right leg: base change exchanges the middle pullback and pushforward in a composite, while compatibility with equivalences makes the result independent of the representative span. Here the target is $\mathrm{Vect}_{\mathbb{Q}}$, so these equalities suffice; no adjunction is required. The external product $(\varphi \boxtimes \psi)(x, y) = \varphi(x)\psi(y)$ and $D(*) = \mathbb{Q}$ make it lax symmetric monoidal, by multiplicativity of homotopy cardinality [GKT18b, Corollary 3.6]. It need not be strong: on infinite sets of components, not every function on a product is a finite sum of external products. For groupoids, the fibre weight is $1/|\operatorname{Aut}_{X_y}(z)|$; for a discrete fibre, it is 1.

To apply this to a 2-Segal space, it suffices that X_1 be locally finite and that $s_0\colon X_0 \to X_1$ and $d_1\colon X_2 \to X_1$ be finite. Then all X_n are locally finite and the multiplication and unit spans are admissible [GKT18a, Section 7.4 and Lemma 7.5].

Example 2.12 (The finite-field Hall product)**.** Let $\mathcal{A} = \mathrm{Rep}_{\mathbb{F}_q}(Q)$ for a finite quiver Q, and take $X_\bullet = S_\bullet(\mathcal{A})^{\simeq}$. The groupoid $X_1 = \mathcal{A}^{\simeq}$ is locally finite because the automorphism group of each representation is finite. The unit map is finite, since $X_0 \simeq *$ represents the zero object. An object of the fibre of d_1 over E is an exact sequence together with an identification of its middle term with E. Such a sequence is determined, up to unique isomorphism over E, by its subrepresentation $N' \subset E$; the quotient is then E/N'. An automorphism fixing E fixes both end terms, so this fibre is the discrete set of subrepresentations of E. This set is finite over $\mathbb{F}_q$, so d_1 is finite and every fibre weight equals 1. Hence the realization gives

$$(d_1)_!(d_0, d_2)^*(f \boxtimes g)(E) = \sum_{N'\subset E} f(E/N')g(N').$$

The input map (d_0, d_2) also has finite fibres, ensuring that finitely supported functions form a subalgebra [GKT18a, Section 7.16]. This is the classical Hall algebra of Section 1, with basis δ_M and structure constants $F^E_{M,N}$.

3 Further examples

We now examine four aspects of the construction: the obstruction to bialgebra compatibility, the geometry behind the quantum Serre relations, the local description of the Jordan Hall algebra, and coherent actions on Hilbert schemes. These examples connect the span formalism with counting, constructible sheaves and derived categories.

3.1 The Hall coproduct and the Euler twist

Let $\mathcal{A} = \mathrm{Rep}_{\mathbb{F}_q}(Q)$ for a finite acyclic quiver, and put $X_\bullet = S_\bullet(\mathcal{A})^{\simeq}$. Reversing the extension span gives a comultiplication in spans and its counting realization:

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_0,d_2)} X_1^2, \qquad \Delta_0[E] = \sum_{[M],[N]} \frac{a_M a_N}{a_E} F^E_{M,N}[M] \otimes [N],$$

where $a_M = |\mathrm{Aut}(M)|$. The fibre over fixed end terms (M, N) has extension classes as components and $\mathrm{Hom}(M, N)$ as the automorphism group of each extension. Thus its contribution with middle term E is

$$\frac{|\mathrm{Ext}^1(M,N)_E|}{|\mathrm{Hom}(M,N)|} = \frac{a_M a_N}{a_E} F^E_{M,N}.$$

The sum is finite because E has finitely many subrepresentations. The same flag comparisons as before give coassociativity, and the counit is $\varepsilon([E]) = \delta_{E,0}$.

The obstruction to a bialgebra already appears before counting, as explained in [Dyc15, Section 2.4]. Write m and Δ for the multiplication and comultiplication spans. The middle groupoid of Δm consists of two exact sequences with the same middle term, whereas that of

$$(m \otimes m)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)$$

consists of four exact sequences whose corner terms agree; here τ exchanges the two middle factors. To compare them, consider commutative diagrams with all rows and columns short exact, omitting their zero terms:

$$\begin{array}{ccccc}
A & \rightarrowtail & B & \twoheadrightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
A' & \rightarrowtail & E & \twoheadrightarrow & C' \\
\downarrow & & \downarrow & & \downarrow \\
A'' & \rightarrowtail & B'' & \twoheadrightarrow & C''.
\end{array}$$

Call the groupoid of such diagrams Sq, the middle row and column Cross, and the four outer exact sequences Frame. In our quotient-first order, both composite spans have inputs

(B'', B) and outputs (C', A'). A cross determines its corners: $A = B \times_E A'$, $C'' = E/(B + A')$, and the remaining corners are the corresponding quotients. Hence forgetting the corners gives an equivalence, but forgetting the centre need not:

$$\mathrm{Cross} \xleftarrow{\sim} \mathrm{Sq} \xrightarrow{\pi} \mathrm{Frame}\,.$$

More explicitly, the square

$$\begin{array}{ccc} \mathrm{Sq} & \xrightarrow{\text{top and bottom}} & X_2 \times X_2 \\ {\scriptstyle \text{left and right}}\downarrow & & \downarrow \\ X_2 \times X_2 & \longrightarrow & X_1^4 \end{array}$$

records the four corners in the order (A, C, A'', C''). Its pullback is Frame, so the comparison map to that pullback is π. This is the square that would have to be Cartesian to deduce ordinary bialgebra compatibility by base change.

For a fixed frame, filling in E is a trivialization problem for a two-extension of C'' by A. In the hereditary case the Ext^2 obstruction vanishes, but the filling groupoid is still not generally contractible: its isomorphism classes form a torsor for $\mathrm{Ext}^1(C'', A)$, and each filling has automorphism group $\mathrm{Hom}(C'', A)$ [Dyc15, Lemmas 2.26, 2.30 and Proposition 2.38]. For example, in vector spaces, take $A = C'' = \mathbb{F}_q$ and $C = A'' = 0$, with the evident outer exact sequences. Filling the centre amounts to choosing $0 \to \mathbb{F}_q \to E \to \mathbb{F}_q \to 0$. There is one isomorphism class of fillings, but its automorphism group fixing the frame is $(\mathbb{F}_q, +)$. The fibre is therefore $B(\mathbb{F}_q, +)$, not a point.

With $\langle M, N\rangle = \dim \mathrm{Hom}(M, N) - \dim \mathrm{Ext}^1(M, N)$, counting this defect gives the Euler factor

$$|\pi^{-1}(f)| = \frac{|\,\mathrm{Ext}^1(C'', A)|}{|\,\mathrm{Hom}(C'', A)|} = q^{-\langle C'', A\rangle}.$$

Thus the multiplication and coproduct do not in general form an ordinary bialgebra, and hence do not give an ordinary Hopf algebra with these operations. Green's formula corrects their compatibility by the Euler form [Dyc15, Section 2.4.3]. In our convention, with $v^2 = q$, use $[M] \star [N] = v^{\langle M,N\rangle}[M][N]$ and

$$\Delta_v[E] = \sum_{[M],[N]} v^{\langle M,N\rangle} \frac{a_M a_N}{a_E} F^E_{M,N}\,[M] \otimes [N].$$

Then Δ_v is an algebra homomorphism for the twisted tensor-product multiplication

$$(a \otimes b) \star (c \otimes d) = v^{\langle |b|,|c|\rangle + \langle |c|,|b|\rangle}(a \star c) \otimes (b \star d),$$

where $|a|$ denotes the dimension-vector degree of a homogeneous element [BK01, Section 1.4]. The Euler form thus controls both the quantum Serre relations and the compatibility of multiplication with comultiplication. Adjoining the Cartan generators yields the usual extended Hall Hopf algebra [Sch06, Sections 1.5–1.7].

3.2 Perverse sheaves and the quantum Serre relations

Let Q be a finite quiver without loops and work over $\mathbb{C}$. Let $\mathfrak{g}_Q$ be the symmetric Kac–Moody algebra attached to its underlying graph. Representations of dimension vector $\mathbf{d}$ form

$$R_{\mathbf{d}} = \bigoplus_{a\colon i\to j} \operatorname{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}), \qquad G_{\mathbf{d}} = \prod_i \mathrm{GL}_{d_i},$$

with change-of-basis action by $G_{\mathbf{d}}$. The extension correspondence now acts on equivariant constructible complexes by induction. Its generating inputs are the constant sheaves at the vertex-simple representations S_i. For a word $\mathbf{i} = (i_1, \dots, i_r)$, let $\pi_{\mathbf{i}}\colon \widetilde{R}_{\mathbf{i}} \to R_{\mathbf{d}}$ forget a representation-stable flag whose simple quotients, read from the top down, are $S_{i_1}, \dots, S_{i_r}$. The flag incidence variety is smooth and the forgetful map is proper. Its normalized push-forward

$$L_{\mathbf{i}} = (\pi_{\mathbf{i}})_! \mathbb{Q}[\dim \widetilde{R}_{\mathbf{i}}]$$

is a direct sum of shifts of simple perverse sheaves by the decomposition theorem. These summands and their shifts generate Lusztig's additive category; normalized induction realizes concatenation of words [Sch09, Sections 1.3–1.4].

Why does the A_2 calculation control the simply-laced Serre relations? Each defining relation involves only two vertices i, j. Setting all other components of the dimension vector to zero restricts the representation and flag spaces to that two-vertex subquiver. If the vertices are not joined, the two induction operations commute; if they are joined by one edge, the subquiver is A_2. For multiple edges one must instead use the corresponding rank-two relation

$$\sum_{a=0}^{1-c_{ij}} (-1)^a e_i^{(a)} e_j e_i^{(1-c_{ij}-a)} = 0,$$

where c_{ij} is the Cartan entry and $e_i^{(a)} = e_i^a/[a]_v!$, with $[a]_v! = \prod_{s=1}^{a} \frac{v^s - v^{-s}}{v - v^{-1}}$. Thus reduction to A_2 concerns the single-edge relations; proving that these relations give the entire algebra still requires Lusztig's theorem.

The geometry of a single arrow is governed by Grassmannians. For $V = \mathbb{C}^m$, $W = \mathbb{C}^n$, and $0 \le a \le m$, consider

$$Z_a = \{(f, L) \mid f\colon V \to W,\ L \in \mathrm{Gr}(a, V),\ f(L) = 0\}, \qquad \pi_a\colon Z_a \to \operatorname{Hom}(V, W).$$

Over $\mathrm{Gr}(a, V)$, this is the vector bundle with fibre $\operatorname{Hom}(V/L, W)$, so Z_a is smooth. The map π_a is proper, and over a rank-ρ map its fibre is

$$\pi_a^{-1}(f) = \mathrm{Gr}(a, \ker f) \simeq \mathrm{Gr}(a, m - \rho).$$

For $a = m - \rho_0$ with $0 \le \rho_0 \le \min(m, n)$, it resolves the determinantal locus of maps of rank at most ρ_0. Grassmannian cohomology is concentrated in even degrees, with

$$\sum_j \dim H^{2j}(\mathrm{Gr}(a, d), \mathbb{Q})\, t^j = \begin{bmatrix} d \\ a \end{bmatrix}_t.$$

This is the same Gaussian polynomial that occurred in the finite-field subspace count. The Beilinson–Bernstein–Deligne–Gabber decomposition theorem expresses $R(\pi_a)_* \mathbb{Q}[\dim Z_a]$ as

a direct sum of shifted intersection complexes on closures of rank strata [BBD82, Theorem 6.2.5]. The Grassmannian fibres determine its stalk cohomology; the contributions from the different intersection complexes must then be separated to find the summands.

The smallest Serre calculation makes this decomposition explicit. For $1 \to 2$ and dimension vector $(2,1)$, we have $R_{(2,1)} = \mathbb{A}^2$. The two relevant simple perverse sheaves are $\mathrm{IC}_{\mathbb{A}^2} = \mathbb{Q}_{\mathbb{A}^2}[2]$ and the skyscraper sheaf IC_0 at the zero representation. The flag spaces for 112, 211, and 121 are respectively

$$\mathbb{A}^2 \times \mathbb{P}^1, \qquad \{0\} \times \mathbb{P}^1, \qquad \{(f, L) \mid L \subset \ker f\} \simeq \mathrm{Bl}_0\, \mathbb{A}^2.$$

The first projection is a $\mathbb{P}^1$-bundle, the second has fibre $\mathbb{P}^1$ over the origin, and the third is an isomorphism away from the origin with exceptional fibre $\mathbb{P}^1$. Their pushforwards give

$$\begin{aligned} L_{112} &\simeq \mathrm{IC}_{\mathbb{A}^2}[1] \oplus \mathrm{IC}_{\mathbb{A}^2}[-1], \\ L_{211} &\simeq \mathrm{IC}_0[1] \oplus \mathrm{IC}_0[-1], \\ L_{121} &\simeq \mathrm{IC}_{\mathbb{A}^2} \oplus \mathrm{IC}_0\,. \end{aligned}$$

Consequently,

$$L_{112} \oplus L_{211} \simeq L_{121}[1] \oplus L_{121}[-1].$$

Let x and y be the classes of the simple-input sheaves at vertices 1 and 2. In the split Grothendieck group, with $[K[1]] = v[K]$, this becomes

$$x^2y - (v + v^{-1})xyx + yx^2 = 0.$$

The calculation for dimension vector $(1,2)$ gives the other Serre relation. Thus the coefficient $v + v^{-1}$ comes from the two cohomological degrees of $\mathbb{P}^1$. Lusztig's theorem identifies the resulting graded Grothendieck algebra, after extending scalars to $\mathbb{Q}(v)$, with $U_v^+(\mathfrak{g}_Q)$; the simple perverse summands give its canonical basis [Sch09, Section 2.2 and Theorem 3.6]. Here we use the split group of the additive category, so shifts retain the parameter v; the ordinary triangulated Grothendieck group would instead identify a shift with a minus sign.

The same incidence spaces also occur in Jiang's theorem on derived categories of derived Grassmannians. Assume $m \geq n$, put $r = m - n$, and on $X = \mathrm{Hom}(V, W)$ take the universal two-term complex

$$\mathcal{K} = [W^\vee \otimes \mathcal{O}_X \xrightarrow{f^\vee} V^\vee \otimes \mathcal{O}_X]$$

in cohomological degrees $(-1, 0)$. A rank-a locally free quotient of $\mathcal{K}$ is a quotient of $V^\vee$ annihilating $f^\vee$, equivalently an a-plane in $\ker f$. Thus $\mathrm{Gr}_X(\mathcal{K}; a)$ is exactly Z_a. The dual derived Grassmannian $\mathrm{Gr}_X(\mathcal{K}^\vee[1]; b)$ is the companion incidence space

$$Z_b^- = \{(f, W \xrightarrow{q} T) \mid q \text{ surjective},\ \dim T = b,\ qf = 0\}.$$

In this universal setting both spaces are smooth classical schemes, but their common incidence correspondences retain the derived structure needed for the functors. Jiang's theorem gives a semiorthogonal decomposition of $\mathrm{Perf}(Z_a)$ into fully faithful images of

$$\mathrm{Perf}(Z_{a-i}^-), \qquad 0 \leq i \leq \min(r, a),$$

with $\binom{r}{i}$ copies for each i, indexed by partitions in the $(r-i)$-by-i rectangle and ordered as in the theorem [Jia23, Notation 3.1 and Theorem 3.2]. The general theorem applies to

a perfect complex of homological Tor-amplitude $[0,1]$ and nonnegative virtual rank on a prestack over $\mathbb{Q}$, and constructs the embeddings by derived incidence kernels.

For $m=2$, $n=a=1$, these spaces are $Z_1=\mathrm{Bl}_0\,\mathbb{A}^2$, $Z_1^-=\{0\}$, and $Z_0^-=\mathbb{A}^2$. The formula recovers the familiar blowup decomposition

$$D^b\,\mathrm{Coh}(\mathrm{Bl}_0\,\mathbb{A}^2)=\left\langle i_*\mathcal{O}_{\mathbb{P}^1}(-1),\ L\pi_1^*D^b\,\mathrm{Coh}(\mathbb{A}^2)\right\rangle,$$

where i is the exceptional divisor. This is the coherent-category counterpart of the same geometry that gave $R(\pi_1)_*\mathbb{Q}[2]\simeq \mathrm{IC}_{\mathbb{A}^2}\oplus \mathrm{IC}_0$ above. The BBDG statement is a direct-sum decomposition of constructible complexes, while Jiang's statement is a semiorthogonal decomposition of coherent categories; the connection is the shared Grassmannian incidence geometry, not a termwise identification of the two decompositions [Jia23, Section 1.2]. There is also a direct Hall-theoretic bridge: Toda proves the Quot decomposition formula using categorified Hall products for Grassmannian flips [Tod23, Theorem 1.1].

3.3 The Jordan quiver and torsion sheaves on $\mathbb{A}^1$

The affine line is the natural geometric setting for the Jordan quiver J:

$$\mathrm{Rep}_k(J)\simeq \mathrm{Mod}^{\mathrm{fd}}(k[t])\simeq \mathrm{Coh}_0(\mathbb{A}^1_k).$$

The endomorphism of a representation is multiplication by the coordinate t. Nilpotent representations correspond to sheaves supported at the origin; allowing arbitrary endomorphisms allows all finite supports on $\mathbb{A}^1_k$.

Over $k=\mathbb{F}_q$, every torsion sheaf has a canonical decomposition

$$F=\bigoplus_{x\in|\mathbb{A}^1_k|}F_x$$

with only finitely many nonzero summands, where x runs over all closed points. At a point of degree d_x, the residue field is $k(x)=\mathbb{F}_{q^{d_x}}$. Choosing a local parameter identifies the completed local ring with $k(x)[[t]]$, so sheaves supported at x are nilpotent Jordan representations over $k(x)$. Their indecomposables are $J_{r,x}=\mathcal{O}_{\mathbb{A}^1,x}/\mathfrak{m}_x^r$, of local length r and k-dimension rd_x. In particular, for $S_x=J_{1,x}$, the earlier calculation reads

$$[S_x]^2=[J_{2,x}]+(q^{d_x}+1)[S_x\oplus S_x].$$

There are no Hom or Ext groups between sheaves with disjoint support, so their Hall product is simply their direct sum. It follows that

$$\mathcal{H}(\mathrm{Coh}_0(\mathbb{A}^1_k))\simeq \bigotimes_{x\in|\mathbb{A}^1_k|}{}'\,\mathcal{H}(\mathrm{Rep}^{\mathrm{nil}}_{k(x)}(J)),$$

where the restricted tensor product allows only finitely many nonunit factors [BK01, Sections 2.4 and 3.3]. The Hall algebra of torsion sheaves on $\mathbb{A}^1$ is therefore assembled from the local symmetric-function algebras, with Hall parameter q^{d_x} at x.

3.4 Zero-dimensional sheaves and Hilbert schemes

The Hall action in this example comes from a *Segal module*, expressed here as a relative 2-Segal object. This supplies a module object over the Hall $\mathbb{E}_1$-algebra, including all higher associativity and unit compatibilities, before passing to any numerical invariant [God25, Corollary 5.4]. We describe the geometric data and the resulting action; a proof of the general module theorem is not needed here. Let S be a smooth surface over a field k. Zero-dimensional coherent sheaves form an extension-closed abelian subcategory of $\mathrm{Coh}(S)$, so their S-construction gives a Hall algebra object in spans. Its degree-one stack $\mathrm{Coh}_0(S)$ parametrizes T-flat coherent sheaves on $S \times_k T$ with support finite over T. Write $\mathrm{Coh}_0^{(m)}(S)$ for the length-m component.

The Hilbert scheme $S^{[n]}$ parametrizes subschemes $Z \subset S$ of length n, equivalently ideals $I_Z \subset \mathcal{O}_S$ of colength n. For a test scheme T, its T-points are ideals $\mathcal{I} \subset \mathcal{O}_{S\times_k T}$ such that $\mathcal{O}_{S\times_k T}/\mathcal{I}$ is finite and flat of rank n over T. Although ideal sheaves are not extension-closed, inclusions of ideals with zero-dimensional quotients define modifications between Hilbert schemes. These will make $\mathcal{H} = \coprod_{n\geq 0} S^{[n]}$ a module over the Hall object of zero-dimensional sheaves.

If $Z \subset Z'$ have lengths n and $n+m$, set $Q = I_Z/I_{Z'}$, of length m. The inclusions $I_{Z'} \subset I_Z \subset \mathcal{O}_S$ give

$$0 \longrightarrow I_{Z'} \longrightarrow I_Z \longrightarrow Q \longrightarrow 0,$$
$$0 \longrightarrow Q \longrightarrow \mathcal{O}_{Z'} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

The nested Hilbert scheme $S^{[n,n+m]}$ gives the action span

$$\mathrm{Coh}_0^{(m)}(S) \times S^{[n+m]} \xleftarrow{p} S^{[n,n+m]} \xrightarrow{q} S^{[n]},$$

where $p(Z \subset Z') = (I_Z/I_{Z'}, Z')$ and $q(Z \subset Z') = Z$. Thus Q acts on Z' to produce Z, lowering its length by m.

For two successive modifications $Z \subset Z' \subset Z''$, the quotients $Q_2 = I_{Z'}/I_{Z''}$ and $Q_1 = I_Z/I_{Z'}$ fit into

$$0 \longrightarrow Q_2 \longrightarrow I_Z/I_{Z''} \longrightarrow Q_1 \longrightarrow 0.$$

Acting first by Q_2, then by Q_1, therefore combines them in the quotient-first order $[Q_1][Q_2]$.

To express this compatibility for all lengths, let Y_r parametrize chains

$$Z_0 \supset Z_1 \supset \cdots \supset Z_r \qquad (I_{Z_0} \subset I_{Z_1} \subset \cdots \subset I_{Z_r}),$$

so $Y_0 = \mathcal{H}$ and Y_1 is the disjoint union of the nested Hilbert schemes. The map to the degree-r stack $S_r\,\mathrm{Coh}_0(S)$ of exact flags records

$$0 \subset I_{Z_1}/I_{Z_0} \subset \cdots \subset I_{Z_r}/I_{Z_0}.$$

A chain is determined by its successive arrows, which is the 1-Segal condition. The relative comparison amounts to inserting intermediate ideals: a subobject of I_{Z_j}/I_{Z_i} lifts uniquely by inverse image to an ideal between I_{Z_i} and I_{Z_j}.

This also works for families. Given Hilbert families $I \subset J \subset \mathcal{O}_{S\times_k T}$, put $Q = J/I$ and take an admissible subobject $A \subset Q$ in the exact category of T-flat zero-dimensional sheaves. The inverse image $K \subset J$ of A fits into an exact sequence

$$0 \longrightarrow Q/A \longrightarrow \mathcal{O}_{S\times_k T}/K \longrightarrow \mathcal{O}_{S\times_k T}/J \longrightarrow 0.$$

The quotient Q/A is T-flat by the choice of exact structure. Both end terms are therefore T-flat with finite support over T, so $\mathcal{O}_{S\times_k T}/K$ defines an intermediate Hilbert family. Flatness also makes the construction compatible with base change. These compatible reconstructions are the *relative 2-Segal* condition for $Y_\bullet \to S_\bullet \mathrm{Coh}_0(S)$. Just as the 2-Segal condition supplies coherent multiplication, its relative version supplies a coherent action on $Y_0 = \mathcal{H}$ [You18, Theorem 4.2]. With the quotient-first convention used here, this is a left module in spans: two successive modifications combine by the product $[Q_1][Q_2]$ computed above [God25, Remark 2.17 and Corollary 5.4]. A linearization for which the action span is admissible then makes $D(\mathcal{H})$ a module over the realized Hall algebra.

Now let S be a smooth projective complex surface. A Borel–Moore realization of this correspondence would give operators

$$H_*^{\mathrm{BM}}(\mathrm{Coh}_0^{(m)}(S)) \otimes H_*^{\mathrm{BM}}(S^{[n+m]}) \longrightarrow H_*^{\mathrm{BM}}(S^{[n]}).$$

Such a realization requires a derived Hecke correspondence with a refined pullback and a proper pushforward. Diaconescu–Porta–Sala construct the categorical module on the derived Hilbert stack $\mathfrak{Hilb}_r(S) \simeq S^{[r]} \times B\mathbb{G}_{\mathrm{m}}$, together with its weight-graded form and an analogous motivic Borel–Moore action [DPS25, Theorem III.2.5, Remark III.2.6 and Corollary III.2.7]. Passing from their stack-valued construction to the displayed operators on ordinary Hilbert schemes requires compatibility of the refined pullback and pushforward with rigidification; that comparison remains to be supplied here.

3.5 Recent developments and outlook

For further developments, see the work of Diaconescu–Porta–Sala–Schiffmann–Vasserot on nilpotent cohomological Hall algebras of surfaces and their relations with affine Yangians [Dia+26b; Dia+26a].

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