Spherical Functors and $\mathbb{P}^n$-Functors

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Scope and source

This standalone note collects the categorical foundations in the live spherical-adjunction chapter and the independent $\mathbb{P}^n$-functor appendix. It preserves the definitions, conventions, formal criteria, diagrams, proofs, and citations needed for a coherent reading. The derived-Grassmannian chapters and their applications are intentionally excluded. The Anno–Logvinenko autoequivalence theorem remains a cited theorem, as in the source appendix; its direct proof is not reproduced there.

Chapter 1

Spherical adjunctions and categorical twists

This chapter reviews spherical adjunctions, a standard source of nontrivial autoequivalences of derived categories and a categorical B-model counterpart of Dehn twists on the A-model side. Our treatment follows [AL15] and uses stable ∞-categories to make the fiber and cofiber constructions canonical. The independent discussion of $\mathbb{P}^n$-functors appears in an appendix.

1.1 Adjunctions and spherical twists

We begin with several standard facts about adjunctions of ∞-categories; see [Lur26, Tag 02C9].

Definition 1.1.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories. We say that F is left adjoint to G if there exist a unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ and a counit $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ satisfying the triangle identities:

- The composite
$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$$
is id_F in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.
- The composite
$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$$

is id_G in the ∞-category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$.

Spherical adjunction.

Definition 1.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of stable ∞-categories with left and right adjoints $L, R\colon \mathcal{D} \to \mathcal{C}$. The four associated functors are

$$\begin{aligned} T_{\mathcal{C}} &= \mathrm{cofib}(\mathrm{id}_{\mathcal{C}} \to RF), & T_{\mathcal{D}} &= \mathrm{fib}(FR \to \mathrm{id}_{\mathcal{D}}), \\ T'_{\mathcal{C}} &= \mathrm{fib}(LF \to \mathrm{id}_{\mathcal{C}}), & T'_{\mathcal{D}} &= \mathrm{cofib}(\mathrm{id}_{\mathcal{D}} \to FL). \end{aligned}$$

For $L \dashv F \dashv R$, write

$$\eta_L\colon \mathrm{id}_{\mathcal{D}} \to FL, \qquad \varepsilon_L\colon LF \to \mathrm{id}_{\mathcal{C}}$$

for the unit and counit of $L \dashv F$, and

$$\eta_R\colon \mathrm{id}_{\mathcal{C}} \to RF, \qquad \varepsilon_R\colon FR \to \mathrm{id}_{\mathcal{D}}$$

for those of $F \dashv R$.

Definition 1.1.3. An adjoint triple

$$L \dashv F \dashv R$$

between stable ∞-categories is a *spherical adjunction* if the source twist $T_{\mathcal{C}}$ and the target cotwist $T_{\mathcal{D}}$ are equivalences. In this case, we also call F a *spherical functor*.

Lemma 1.1.4. *Let $F\colon \mathcal{C} \to \mathcal{D}$ have right adjoint R. There are natural equivalences*

$$FT_{\mathcal{C}} \simeq T_{\mathcal{D}}F, \qquad T_{\mathcal{C}}R \simeq RT_{\mathcal{D}}.$$

Proof. It suffices to prove $FT_{\mathcal{C}} \simeq T_{\mathcal{D}}F$; the other equivalence is analogous. The defining fiber and cofiber sequences form the pushout diagram

$$\begin{array}{ccccc} F \circ \mathrm{id}_{\mathcal{C}} & \longrightarrow & FRF & \longrightarrow & \mathrm{id}_{\mathcal{D}} \circ F \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F \circ T_{\mathcal{C}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & T_{\mathcal{D}} \circ F[1] \end{array},$$

where 0 is the zero object and each square is a pushout. We therefore obtain the natural equivalence

$$FT_{\mathcal{C}} \simeq T_{\mathcal{D}}F.$$

□

Lemma 1.1.5. *For an adjoint triple $L \dashv F \dashv R$, there are adjunctions*

$$T'_{\mathcal{D}} \dashv T_{\mathcal{D}}, \qquad T'_{\mathcal{C}} \dashv T_{\mathcal{C}}.$$

Proof. It suffices to prove $T'_{\mathcal{D}} \dashv T_{\mathcal{D}}$; the other adjunction is analogous. By the mapping-space criterion for adjunctions [Lur26, Corollary 6.2.6.5], it suffices to construct a natural equivalence

$$\mathrm{Map}_{\mathcal{D}}(M, T_{\mathcal{D}}N) \simeq \mathrm{Map}_{\mathcal{D}}(T'_{\mathcal{D}}M, N)$$

for all $M, N \in \mathcal{D}$. The defining fiber and cofiber sequences give

$$\begin{aligned}
\mathrm{Map}_{\mathcal{D}}(M, T_{\mathcal{D}}N) &\simeq \mathrm{fib}(\mathrm{Map}_{\mathcal{D}}(M, FR(N)) \to \mathrm{Map}_{\mathcal{D}}(M, N)) \\
&\simeq \mathrm{fib}(\mathrm{Map}_{\mathcal{C}}(L(M), R(N)) \to \mathrm{Map}_{\mathcal{D}}(M, N)) \\
&\simeq \mathrm{fib}(\mathrm{Map}_{\mathcal{D}}(FL(M), N) \to \mathrm{Map}_{\mathcal{D}}(M, N)) \\
&\simeq \mathrm{Map}_{\mathcal{D}}(T'_{\mathcal{D}}M, N).
\end{aligned}$$

Here postcomposition with $\varepsilon_R(N)$ corresponds, under the two adjunction equivalences, to precomposition with $\eta_L(M)$. □

We next construct the unit and counit of $T'_{\mathcal{D}} \dashv T_{\mathcal{D}}$. Write the defining sequences as

$$\mathrm{id}_{\mathcal{D}} \xrightarrow{\eta_L} FL \xrightarrow{q} T'_{\mathcal{D}}, \qquad T_{\mathcal{D}} \xrightarrow{i} FR \xrightarrow{\varepsilon_R} \mathrm{id}_{\mathcal{D}}\,.$$

Since $\mathcal{D}$ is stable, the first sequence identifies $\mathrm{id}_{\mathcal{D}}$ with the fiber of q, and the second identifies $T_{\mathcal{D}}$ with the fiber of ε_R.

Constructing the unit. To construct the unit, it is enough to compare the arrow $q\colon FL \to T'_{\mathcal{D}}$ with $\varepsilon_R T'_{\mathcal{D}}\colon FRT'_{\mathcal{D}} \to T'_{\mathcal{D}}$. Define

$$f : FL \xrightarrow{F\eta_R L} FRFL \xrightarrow{FRq} FRT'_{\mathcal{D}}.$$

The identity $\varepsilon_R T'_{\mathcal{D}} \circ f = q$, proved below, gives a morphism of fiber sequences

$$\begin{array}{ccccc} \mathrm{id}_{\mathcal{D}} & \xrightarrow{\eta_L} & FL & \xrightarrow{q} & T'_{\mathcal{D}} \\ {\scriptstyle \exists \eta}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} \\ T_{\mathcal{D}} T'_{\mathcal{D}} & \xrightarrow[iT'_{\mathcal{D}}]{} & FRT'_{\mathcal{D}} & \xrightarrow[\varepsilon_R T'_{\mathcal{D}}]{} & T'_{\mathcal{D}} \end{array}$$

and the induced map on fibers is the unit $\eta\colon \mathrm{id}_{\mathcal{D}} \to T_{\mathcal{D}} T'_{\mathcal{D}}$.

Naturality of $\varepsilon_R\colon FR \to \mathrm{id}_{\mathcal{D}}$ with respect to $q\colon FL \to T'_{\mathcal{D}}$ gives the commutative square

$$\begin{array}{ccc} FRFL & \xrightarrow{FRq} & FRT'_{\mathcal{D}} \\ {\scriptstyle \varepsilon_R FL}\downarrow & & \downarrow{\scriptstyle \varepsilon_R T'_{\mathcal{D}}} \\ FL & \xrightarrow[q]{} & T'_{\mathcal{D}} \end{array}$$

Therefore

$$\varepsilon_R T'_{\mathcal{D}} \circ f = q \circ \varepsilon_R FL \circ F\eta_R L.$$

The last two factors are the triangle identity $\varepsilon_R F \circ F\eta_R = \mathrm{id}_F$ for $F \dashv R$, whiskered by L. Hence

$$\varepsilon_R FL \circ F\eta_R L = \mathrm{id}_{FL},$$

and therefore $\varepsilon_R T'_{\mathcal{D}} \circ f = q$.

Constructing the counit. For the counit, we compare the cofiber sequence defining $T'_{\mathcal{D}} T_{\mathcal{D}}$ with the sequence defining $\mathrm{id}_{\mathcal{D}}$ as the cofiber of i. Define

$$g : FLT_{\mathcal{D}} \xrightarrow{FLi} FLFR \xrightarrow{F\varepsilon_L R} FR.$$

The identity $g \circ \eta_L T_{\mathcal{D}} = i$ makes the left square below commute, so the universal property of the cofiber gives the counit $\varepsilon\colon T'_{\mathcal{D}} T_{\mathcal{D}} \to \mathrm{id}_{\mathcal{D}}$:

$$\begin{array}{ccccc} T_{\mathcal{D}} & \xrightarrow{\eta_L T_{\mathcal{D}}} & FLT_{\mathcal{D}} & \xrightarrow{qT_{\mathcal{D}}} & T'_{\mathcal{D}} T_{\mathcal{D}} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \exists \varepsilon} \\ T_{\mathcal{D}} & \xrightarrow[i]{} & FR & \xrightarrow[\varepsilon_R]{} & \mathrm{id}_{\mathcal{D}} \end{array}$$

Indeed, naturality of η_L with respect to $i\colon T_{\mathcal{D}} \to FR$ gives

$$FLi \circ \eta_L T_{\mathcal{D}} = \eta_L FR \circ i.$$

Thus

$$g \circ \eta_L T_{\mathcal{D}} = F\varepsilon_L R \circ \eta_L FR \circ i = i,$$

where the last equality is the triangle identity $F\varepsilon_L \circ \eta_L F = \mathrm{id}_F$ for $L \dashv F$, whiskered by R.

Theorem 1.1.6. *The maps η and ε satisfy the triangle identities and define the adjunction $T'_{\mathcal{D}} \dashv T_{\mathcal{D}}$.*

Proof. We verify the first triangle identity; the second follows from the dual argument in $\mathcal{D}^{\mathrm{op}}$. Stack the unit construction, whiskered by $T_{\mathcal{D}}$, above the counit construction:

$$\begin{array}{ccccc}
T_{\mathcal{D}} & \longrightarrow & FLT_{\mathcal{D}} & \longrightarrow & T'_{\mathcal{D}}T_{\mathcal{D}} \\
{\scriptstyle \eta T_{\mathcal{D}}}\downarrow & & \downarrow{\scriptstyle fT_{\mathcal{D}}} & & \downarrow{\scriptstyle \mathrm{id}} \\
T_{\mathcal{D}}T'_{\mathcal{D}}T_{\mathcal{D}} & \longrightarrow & FRT'_{\mathcal{D}}T_{\mathcal{D}} & \longrightarrow & T'_{\mathcal{D}}T_{\mathcal{D}} \\
{\scriptstyle T_{\mathcal{D}}\varepsilon}\downarrow & & \downarrow{\scriptstyle FR\varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\
T_{\mathcal{D}} & \longrightarrow & FR & \longrightarrow & \mathrm{id}_{\mathcal{D}}
\end{array}$$

It suffices to identify the composite in the middle column. Using $f = FRq \circ F\eta_R L$, we obtain

$$FR\varepsilon \circ fT_{\mathcal{D}} = FR(\varepsilon \circ qT_{\mathcal{D}}) \circ F\eta_R LT_{\mathcal{D}}.$$

By the square defining the counit, $\varepsilon \circ qT_{\mathcal{D}} = \varepsilon_R \circ g$, so

$$FR\varepsilon \circ fT_{\mathcal{D}} = FR\varepsilon_R \circ FRg \circ F\eta_R LT_{\mathcal{D}}.$$

Write $g = Fh$, where $h = \varepsilon_L R \circ Li \colon LT_{\mathcal{D}} \to R$. Naturality of η_R with respect to h gives

$$FRg \circ F\eta_R LT_{\mathcal{D}} = F(\eta_R R \circ h).$$

Therefore

$$FR\varepsilon \circ fT_{\mathcal{D}} = F(R\varepsilon_R \circ \eta_R R) \circ g = g,$$

by the triangle identity for $F \dashv R$. Thus the middle column is the map g used to define the counit. The universal property of the fiber defining $T_{\mathcal{D}}$ identifies the left column with $\mathrm{id}_{T_{\mathcal{D}}}$. Hence

$$(T_{\mathcal{D}}\varepsilon) \circ (\eta T_{\mathcal{D}}) = \mathrm{id}_{T_{\mathcal{D}}}.$$

□

1.2 The 2-out-of-4 criterion

The preceding adjunction yields the following criterion for sphericality.

Theorem 1.2.1 (2-out-of-4)**.** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of stable ∞-categories with left and right adjoints L and R. If any two of the following conditions hold, then all four hold:*

1. *The target cotwist $T_{\mathcal{D}}$ is an equivalence, or equivalently, the target dual twist $T'_{\mathcal{D}}$ is an equivalence.*
2. *The source twist $T_{\mathcal{C}}$ is an equivalence, or equivalently, the source dual cotwist $T'_{\mathcal{C}}$ is an equivalence.*
3. *The comparison map*
$$R \longrightarrow RFL \longrightarrow T_{\mathcal{C}}L$$
is an equivalence, or equivalently, so is
$$L \longrightarrow RFL \longrightarrow RT'_{\mathcal{D}}.$$
4. *The comparison map*
$$LT_{\mathcal{D}} \longrightarrow LFR \longrightarrow R$$
is an equivalence, or equivalently, so is
$$T'_{\mathcal{C}}R \longrightarrow LFR \longrightarrow L.$$

The following two lemmas identify the cofibers of the comparison maps in conditions (3) and (4).

Lemma 1.2.2. *Set $K = \operatorname{cofib}(R \to T_{\mathcal{C}}L)$. There are natural equivalences*

$$\begin{aligned} K &\simeq \operatorname{cofib}(L \to RT'_{\mathcal{D}}),\\ FK &\simeq \operatorname{cofib}(\mathrm{id}_{\mathcal{D}} \to T_{\mathcal{D}}T'_{\mathcal{D}}),\\ KF &\simeq \operatorname{cofib}(\mathrm{id}_{\mathcal{C}} \to T_{\mathcal{C}}T'_{\mathcal{C}}). \end{aligned}$$

Proof. We directly compare the two maps with common target RFL:

$$L \xrightarrow{\eta_R L} RFL \longrightarrow T_{\mathcal{C}}L, \qquad R \xrightarrow{R\eta_L} RFL \longrightarrow RT'_{\mathcal{D}}.$$

Filling in the induced cofibers gives the diagram

$$\begin{array}{ccccc} R & \longrightarrow & RFL & \longrightarrow & T_{\mathcal{C}}L \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & RT'_{\mathcal{D}} & \longrightarrow & K \\ & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & L[1] \end{array}$$

whose small squares are pushouts. The right-hand part gives $K \simeq \mathrm{cofib}(L \to RT'_{\mathcal{D}})$.

Next apply F to the defining cofiber sequence of K. Using the map

$$f : FL \xrightarrow{F\eta_R L} FRFL \xrightarrow{FRq} FRT'_{\mathcal{D}}$$

from the construction of the unit, the diagram induced by these maps is

$$\begin{array}{ccccc} FL & \longrightarrow & FRT'_{\mathcal{D}} & \longrightarrow & T'_{\mathcal{D}} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & FK & \longrightarrow & \mathrm{id}_{\mathcal{D}}[1] \\ & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & T_{\mathcal{D}}T'_{\mathcal{D}}[1] \end{array}$$

Rotating the right column gives

$$FK \simeq \mathrm{cofib}(\mathrm{id}_{\mathcal{D}} \to T_{\mathcal{D}}T'_{\mathcal{D}}).$$

The third equivalence follows in the same way after composing with F on the right. The relevant diagram is

$$\begin{array}{ccccc} RF & \longrightarrow & T_{\mathcal{C}}LF & \longrightarrow & T_{\mathcal{C}} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & KF & \longrightarrow & \mathrm{id}_{\mathcal{C}}[1] \\ & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & T_{\mathcal{C}}T'_{\mathcal{C}}[1] \end{array}$$

and gives

$$KF \simeq \mathrm{cofib}(\mathrm{id}_{\mathcal{C}} \to T_{\mathcal{C}}T'_{\mathcal{C}}).$$

□

Lemma 1.2.3. *Set $K' = \mathrm{cofib}(LT_{\mathcal{D}} \to R)$. There is a natural equivalence*

$$K' \simeq \mathrm{cofib}(T'_{\mathcal{C}}R \to L).$$

If $A \in \mathcal{D}$ and $T_{\mathcal{D}}(A) = 0$, then $A \simeq FK'(A)$. Similarly, if $B \in \mathcal{C}$ and $T_{\mathcal{C}}(B) = 0$, then $B \simeq K'F(B)$.

Proof. Start with the map

$$\alpha : LT_{\mathcal{D}} \xrightarrow{Li} LFR \xrightarrow{\varepsilon_L R} R,$$

whose cofiber is K'. We compare it with the two defining cofiber sequences

$$LT_{\mathcal{D}} \xrightarrow{Li} LFR \to L, \qquad T'_{\mathcal{C}}R \longrightarrow LFR \xrightarrow{\varepsilon_L R} R,$$

and obtain the diagram

$$\begin{array}{ccccc} LT_{\mathcal{D}} & \longrightarrow & LFR & \longrightarrow & R \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L & \longrightarrow & K' \\ & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & T'_{\mathcal{C}}R[1] \end{array}$$

Thus the right column gives $K' \simeq \mathrm{cofib}(T'_{\mathcal{C}}R \to L)$.

Now let $A \in \mathcal{D}$ and suppose $T_{\mathcal{D}}(A) = 0$. The defining fiber sequence

$$T_{\mathcal{D}}(A) \longrightarrow FR(A) \xrightarrow{\varepsilon_R(A)} A$$

shows that $FR(A) \simeq A$. Since the first term of

$$LT_{\mathcal{D}}(A) \longrightarrow R(A) \longrightarrow K'(A)$$

vanishes, we also have $R(A) \simeq K'(A)$. Therefore $A \simeq FK'(A)$.

Similarly, if $T_{\mathcal{C}}(B) = 0$, then the unit $B \to RF(B)$ is an equivalence. Moreover, the equivalence $T_{\mathcal{D}}F \simeq FT_{\mathcal{C}}$ shows that the first term in

$$LT_{\mathcal{D}}F(B) \longrightarrow RF(B) \longrightarrow K'F(B)$$

vanishes. Hence $K'F(B) \simeq RF(B) \simeq B$. □

Proposition 1.2.4. *If either condition* (1) *or condition* (2) *of Theorem 1.2.1 holds, then condition* (3) *is equivalent to condition* (4).

Proof. Assume first that condition (1) holds, so $T_{\mathcal{D}}$ and $T'_{\mathcal{D}}$ are equivalences. Consider the two natural maps

$$\alpha : LT_{\mathcal{D}} \xrightarrow{Li} LFR \xrightarrow{\varepsilon_L R} R, \qquad \beta : L \xrightarrow{\eta_R L} RFL \xrightarrow{Rq} RT'_{\mathcal{D}}.$$

Here β is the second formulation of condition (3), while α is the first formulation of condition (4). We show that they are identified after composing with $T'_{\mathcal{D}}$.

From the construction of the unit $\eta : \mathrm{id}_{\mathcal{D}} \to T_{\mathcal{D}} T'_{\mathcal{D}}$, we have

$$iT'_{\mathcal{D}} \circ \eta = f \circ \eta_L.$$

Since $f = FRq \circ F\eta_R L = F\beta$, this becomes

$$iT'_{\mathcal{D}} \circ \eta = F\beta \circ \eta_L.$$

Now compute:

$$\begin{aligned} \alpha T'_{\mathcal{D}} \circ L\eta &= \varepsilon_L R T'_{\mathcal{D}} \circ L i T'_{\mathcal{D}} \circ L\eta \\ &= \varepsilon_L R T'_{\mathcal{D}} \circ L(iT'_{\mathcal{D}} \circ \eta) \\ &= \varepsilon_L R T'_{\mathcal{D}} \circ L(F\beta \circ \eta_L) \\ &= \varepsilon_L R T'_{\mathcal{D}} \circ LF\beta \circ L\eta_L \\ &= \beta \circ \varepsilon_L L \circ L\eta_L \\ &= \beta. \end{aligned}$$

The fifth equality is naturality of $\varepsilon_L : LF \to \mathrm{id}_{\mathcal{C}}$, applied to $\beta : L \to RT'_{\mathcal{D}}$; the final equality is the triangle identity for $L \dashv F$. Since $T'_{\mathcal{D}}$ and $L\eta$ are equivalences, α is an equivalence if and only if β is. Thus condition (3) is equivalent to condition (4) whenever condition (1) holds.

If condition (2) holds instead, apply the same argument to $T'_{\mathcal{C}} \dashv T_{\mathcal{C}}$. It identifies the two alternative comparison maps in conditions (3) and (4). The two conditions are therefore equivalent whenever either condition (1) or condition (2) holds. □

Proof of Theorem 1.2.1. Proposition 1.2.4 reduces the mixed cases to

$$(1) + (2) \Longrightarrow (3) + (4), \qquad (3) + (4) \Longrightarrow (1) + (2).$$

Assume first that (3) and (4) hold. Then $K \simeq 0$ and $K' \simeq 0$. The first lemma identifies the unit maps

$$\mathrm{id}_{\mathcal{D}} \to T_{\mathcal{D}} T'_{\mathcal{D}}, \qquad \mathrm{id}_{\mathcal{C}} \to T_{\mathcal{C}} T'_{\mathcal{C}}$$

as equivalences, so $T'_{\mathcal{D}}$ and $T'_{\mathcal{C}}$ are fully faithful. The second lemma shows that $T_{\mathcal{D}}$ and $T_{\mathcal{C}}$ are conservative: if $T_{\mathcal{D}}(A) = 0$, then $A \simeq FK'(A) = 0$, and similarly on $\mathcal{C}$. Thus $T'_{\mathcal{D}} \dashv T_{\mathcal{D}}$ and $T'_{\mathcal{C}} \dashv T_{\mathcal{C}}$ are adjoint equivalences. Hence (1) and (2) hold.

Conversely, assume (1) and (2). The left adjoints $T'_{\mathcal{D}}$ and $T'_{\mathcal{C}}$ are then the respective inverse equivalences, and the first lemma gives

$$FK \simeq 0, \qquad KF \simeq 0.$$

It remains to show that $K \simeq 0$. The unit $L \to LFL$ is a section of the counit $LFL \to L$ by the triangle identity. Hence the fiber of the counit is canonically equivalent to the cofiber of the unit; in the present notation,

$$T'_{\mathcal{C}} L \simeq L T'_{\mathcal{D}}.$$

Equivalently, the two split cofiber sequences

$$T'_{\mathcal{C}} L \to LFL \to L, \qquad L \to LFL \to L T'_{\mathcal{D}}$$

arise from the same canonical splitting and give

$$LFL \simeq L \oplus T'_{\mathcal{C}} L \simeq L \oplus L T'_{\mathcal{D}}.$$

Let

$$T_{\mathcal{C}} L \xrightarrow{\sim} L T_{\mathcal{D}}$$

be the inverse of the double mate of this canonical equivalence. Under this identification, the composite

$$R \xrightarrow{R\eta_L} RFL \to T_{\mathcal{C}} L \simeq L T_{\mathcal{D}} \xrightarrow{Li} LFR \xrightarrow{\varepsilon_L R} R$$

is the identity. Indeed, compatibility of mates with composition identifies its successive mate under $L \dashv F \dashv R$ with

$$L \xrightarrow{L\eta_L} LFL \xrightarrow{\varepsilon_L L} L,$$

which is id_L by the triangle identity. Hence the cofiber sequence

$$R \to T_{\mathcal{C}} L \to K$$

splits, so $T_{\mathcal{C}} L \simeq R \oplus K$.

For any $X \in \mathcal{D}$, we then have

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}}(T_{\mathcal{C}} L(X), K(X)) &\simeq \mathrm{Map}_{\mathcal{C}}(L T_{\mathcal{D}}(X), K(X)) \\ &\simeq \mathrm{Map}_{\mathcal{D}}(T_{\mathcal{D}}(X), FK(X)) \\ &\simeq 0. \end{aligned}$$

Since $K(X)$ is a direct summand of $T_{\mathcal{C}} L(X)$, this forces $K(X) \simeq 0$. Thus $K \simeq 0$, so (3) holds. Finally, (4) follows from Proposition 1.2.4 because $T_{\mathcal{D}}$ is an equivalence. □

1.3 Monadic constructions

Spherical functors are a useful source of autoequivalences of stable ∞-categories in geometry. Conversely, every autoequivalence of a stable ∞-category can be realized as the twist of a spherical functor. The following criterion is the monadic sphericality theorem of [Chr22].

Theorem 1.3.1. *Let $\mathcal{C}$ be a stable ∞-category, and let T be a monad on $\mathcal{C}$ with unit $t\colon \mathrm{id}_{\mathcal{C}} \to T$. Set*

$$T_{\mathcal{C}} = \mathrm{cofib}(\mathrm{id}_{\mathcal{C}} \xrightarrow{t} T).$$

The following conditions are equivalent:

1. *The functor $T_{\mathcal{C}}$ is an equivalence and $T_{\mathcal{C}} t \simeq t T_{\mathcal{C}}$.*
2. *The free–forgetful adjunction*

$$F\colon \mathcal{C} \rightleftarrows \mathrm{LMod}_T(\mathcal{C}) : G$$

is spherical.

Corollary 1.3.2. *Every autoequivalence of a stable ∞-category is the twist of a spherical adjunction.*

Proof. For an autoequivalence T of $\mathcal{C}$, the functor $M = \mathrm{id}_{\mathcal{C}} \oplus T$ has a square-zero extension monad structure. The theorem applies to the free–forgetful adjunction for M, whose source twist is T. □

1.4 Examples

We conclude with several geometric examples. We first record a practical criterion. In this section, $D(X)$ means $D_{\mathrm{qc}}(X)$.

Theorem 1.4.1. *Let X and Y be perfect stacks, and set $\mathcal{C} = D(X)$ and $\mathcal{D} = D(Y)$. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a Fourier–Mukai functor represented by a perfect kernel. If $T_{\mathcal{C}}$ is an autoequivalence, then any equivalence $R \simeq T_{\mathcal{C}} L$ implies that the canonical map in condition* (3) *of Theorem 1.2.1 is an equivalence.*

Proof. Use the spanning class $\Omega = \mathrm{im}(F) \cup \ker R$. It is right spanning. Indeed, suppose that $M \in \mathcal{D}$ is right orthogonal to every object of Ω. Orthogonality to $\mathrm{im}(F)$ gives $R(M) = 0$, so $M \in \ker R \subset \Omega$; testing against M itself gives $M = 0$.

The equivalence of $T_{\mathcal{C}}$ implies $KF = 0$. For an object $M \in \ker R$, consider the fiber sequence

$$R(M) \longrightarrow T_{\mathcal{C}} L(M) \longrightarrow K(M).$$

The equivalence $R \simeq T_{\mathcal{C}} L$ implies $R(M) \simeq T_{\mathcal{C}} L(M) \simeq 0$, so $K(M) \simeq 0$. Thus K vanishes on Ω. By the assumptions, K is again a Fourier–Mukai functor represented by a perfect kernel, and hence admits a right adjoint K^R. For every $N \in \mathcal{C}$ and $M \in \Omega$, adjunction gives

$$\mathrm{Map}_{\mathcal{D}}\big(M, K^R(N)\big) \simeq \mathrm{Map}_{\mathcal{C}}\big(K(M), N\big) \simeq 0.$$

Since Ω is right spanning, $K^R(N) = 0$ for every N, and hence $K = 0$. Therefore the canonical comparison map in condition (3) is an equivalence. □

1. *Classical spherical objects.*

   Let X be a smooth proper variety of dimension n over a field k. A perfect complex $\mathcal{E} \in D(X)$ is spherical if

   $$\mathrm{RHom}_X(\mathcal{E}, \mathcal{E}) \simeq H^*(S^n, \mathbb{Z}) \otimes_{\mathbb{Z}} k \simeq k \oplus k[-n], \qquad \mathcal{E} \otimes \omega_X \simeq \mathcal{E}.$$

   Consider

   $$F = \mathcal{E} \otimes -\colon D(k) \longrightarrow D(X).$$

   Its right adjoint is $R = \mathrm{RHom}_X(\mathcal{E}, -)$, and

   $$RF \simeq \mathrm{RHom}_X(\mathcal{E}, \mathcal{E}) \otimes - \simeq \mathrm{id} \oplus [-n].$$

Hence $T_{\mathcal{C}} \simeq [-n]$. By the Serre functor $S_k \simeq \mathrm{id}$ and $S_X \simeq (-) \otimes \omega_X[n]$, the condition $R \simeq T_{\mathcal{C}} L$ is equivalent to

$$S_X F T_{\mathcal{C}}(-) \simeq (\mathcal{E} \otimes -[-n]) \otimes \omega_X[n] \simeq (\mathcal{E} \otimes \omega_X) \otimes - \simeq F S_k(-).$$

Thus F is spherical by Theorem 1.2.1; its usual twist is the Seidel–Thomas twist [ST01].

2. *EZ-spherical objects.*

   Let $i\colon E \hookrightarrow X$ be a regular embedding of codimension d, and let $q\colon E \to Z$ be flat of relative dimension k, where X, E, Z are smooth and E is proper. Assume that a line bundle θ on Z satisfies

   $$q^*(\theta \otimes \omega_Z^{-1}) \simeq i^* \omega_X^{-1}.$$

   Write $\nu = N_{E/X}$. A perfect complex $K \in D(E)$ is EZ-spherical if, for $d > 0$,

   $$q_* \operatorname{RHom}_E(K, K) \simeq \mathcal{O}_Z, \qquad q_* \operatorname{RHom}_E(K, K \otimes \Lambda^c \nu) \simeq 0 \quad (0 < c < d),$$

   while for $d = 0$ one requires a fiber sequence

   $$\mathcal{O}_Z \longrightarrow q_* \operatorname{RHom}_E(K, K) \longrightarrow \theta[-k].$$

   Then

   $$F_K\colon D(Z) \longrightarrow D(X), \qquad M \longmapsto i_*(K \otimes q^* M),$$

   is spherical [Hor05, Definition 2.1 and Theorem 2.11].

3. *Toda's fat spherical objects.*

   Let X be a smooth projective complex variety of dimension n, A a local Artinian $\mathbb{C}$-algebra, and $\pi\colon X \times \operatorname{Spec} A \to X$ the projection. Let $K \in D(X \times \operatorname{Spec} A)$ be represented by a bounded complex of A-flat coherent sheaves, and set $E = K|_{X \times \{0\}}$. The complex K is A-spherical if

   $$\operatorname{Ext}^i_X(\pi_* K, E) \simeq \begin{cases} \mathbb{C}, & i = 0, n, \\ 0, & \text{otherwise}, \end{cases} \qquad E \otimes \omega_X \simeq E.$$

   Consider the Fourier–Mukai functor

   $$F_K\colon D(\operatorname{Spec} A) \longrightarrow D(X), \qquad M \longmapsto \pi_*(K \otimes \pi'^* M),$$

   where $\pi'\colon X \times \operatorname{Spec} A \to \operatorname{Spec} A$ is the second projection. The functor F_K is spherical, and its twist is an autoequivalence [Tod07, Definition 2.1 and Theorem 2.3].

4. *Hilbert schemes of points on K3 surfaces.*

   Let S be a complex K3 surface, and let $S^{[n]}$ be its Hilbert scheme of length n subschemes. Write $Z \subset S \times S^{[n]}$ for the universal subscheme, with ideal sheaf $\mathcal{I}_Z$, and let
   $$F\colon D(S) \longrightarrow D(S^{[n]})$$
   be the Fourier–Mukai transform with kernel $\mathcal{I}_Z$. Then
   $$RF \simeq \mathrm{id}_S \oplus [-2] \oplus \cdots \oplus [2-2n].$$
   For $n = 2$, the functor F is spherical. For $n > 2$, the same calculation supplies the motivating example of a $\mathbb{P}^{n-1}$-functor rather than a spherical functor; see [Add16, Theorem 2].

5. *Noncommutative K3 surfaces.*

   Let $X \subset \mathbb{P}^5$ be a smooth cubic hypersurface, and let
   $$\mathcal{A} := \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle^{\perp} \subset D(X)$$
   be Kuznetsov's K3 subcategory. Let Y be the variety of lines on X.

   More precisely, let
   $$\mathcal{L} = \{(x, \ell) \in X \times Y \mid x \in \ell\}$$
   be the universal line, with projections $X \xleftarrow{q} \mathcal{L} \xrightarrow{p} Y$. If $I\colon \mathcal{A} \hookrightarrow D(X)$ denotes the inclusion, then the functor
   $$F = p_* q^* I\colon \mathcal{A} \longrightarrow D(Y)$$
   satisfies $RF \simeq \mathrm{id}_{\mathcal{A}} \oplus [-2]$ and is spherical; see [Add16, Theorem 4].

Appendix A

Notes on $\mathbb{P}^n$-functors

Status. This appendix records the construction of $\mathbb{P}^n$-functors and their twists. It is not used elsewhere in the manuscript. The final autoequivalence theorem is quoted from Anno–Logvinenko; its direct proof in the present language is omitted.

Spherical functors provide a categorical framework for Dehn twists and their mirror-symmetric analogues. Their higher-dimensional variants are $\mathbb{P}^n$-functors, introduced as functorial analogues of $\mathbb{P}^n$-objects. Roughly, a functor

$$F\colon \mathcal{C} \longrightarrow \mathcal{D}$$

is a $\mathbb{P}^n$-functor if its adjunction monad behaves like the cohomology algebra of projective space:

$$RF \simeq \mathrm{id}_{\mathcal{C}} \oplus H \oplus H^2 \oplus \cdots \oplus H^n,$$

where R is a right adjoint of F, H is an autoequivalence of $\mathcal{C}$, and multiplication is compatible with the algebra structure of

$$\mathrm{H}^*(\mathbb{P}^n; k) \simeq k[h]/(h^{n+1}).$$

Such a functor determines a twist autoequivalence of $\mathcal{D}$, which recovers the spherical twist when $n = 1$.

A.1 Coherent complexes and convolution

The definition of a $\mathbb{P}^n$-twist uses the convolution of a complex. Given a finite complex in a stable ∞-category,

$$A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_m,$$

one would like to take its "total object", obtained by iteratively forming cofibers. A sequence in the homotopy category alone does not determine such an object canonically: different choices of null-homotopies and higher coherences may lead to nonisomorphic convolutions. A stable ∞-category retains precisely the coherent data needed to make these choices explicit.

The following results are standard; see [Ari21] for complete proofs.

Let $\mathcal{C}$ be a pointed ∞-category. Let Ξ denote the pointed category whose objects are the integers together with a distinguished base point $*$, which is both initial and final. For integers $m, n \in \mathbb{Z}$, the morphisms in Ξ are given by

$$\mathrm{Hom}_{\Xi}(m,n) = \begin{cases} \{*\}, & n \neq m,\ m-1, \\ \{\mathrm{id}_m, *\}, & n = m, \\ \{\partial, *\}, & n = m-1. \end{cases}$$

Here $*$ denotes the zero morphism, and the composition rule implies $\partial \circ \partial = *$ whenever this expression is defined.

Definition A.1.1. A *coherent chain complex* in $\mathcal{C}$ is a pointed functor

$$X_\bullet \colon \Xi \longrightarrow \mathcal{C}.$$

We write

$$\mathrm{Ch}_\bullet(\mathcal{C}) := \mathrm{Fun}_*(\Xi, \mathcal{C})$$

for the ∞-category of coherent chain complexes in $\mathcal{C}$.

Dually, a *coherent cochain complex* in $\mathcal{C}$ is a pointed functor

$$X^\bullet \colon \Xi^{\mathrm{op}} \longrightarrow \mathcal{C},$$

and we write

$$\mathrm{Ch}^\bullet(\mathcal{C}) := \mathrm{Fun}_*(\Xi^{\mathrm{op}}, \mathcal{C}).$$

Now assume that $\mathcal{C}$ is stable. Set

$$\mathrm{Fil}(\mathcal{C}) := \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C}), \qquad \mathrm{gr}(\mathcal{C}) := \mathrm{Fun}(\mathbb{Z}_{\mathrm{disc}}, \mathcal{C}),$$

and let $X \in \mathrm{Fil}(\mathcal{C})$, with stages X^s. Its associated graded pieces are

$$\mathrm{gr}^s X := \mathrm{cofib}\big(X^{s+1} \to X^s\big).$$

For every s, the cofiber sequence

$$\mathrm{gr}^{s+1} X \longrightarrow X^s/X^{s+2} \longrightarrow \mathrm{gr}^s X$$

gives a connecting morphism

$$\partial_s \colon \mathrm{gr}^s X \longrightarrow \mathrm{gr}^{s+1} X[1].$$

The connecting morphisms assemble into the coherent cochain complex

$$\cdots \longrightarrow \mathrm{gr}^{-2} X[-2] \longrightarrow \mathrm{gr}^{-1} X[-1] \longrightarrow \mathrm{gr}^0 X \longrightarrow \mathrm{gr}^1 X[1] \longrightarrow \cdots .$$

We denote this coherent cochain complex by

$$\mathrm{gr}^\bullet X[\bullet] \in \mathrm{Ch}^\bullet(\mathcal{C}).$$

Theorem A.1.2 (Ariotta's E_1-page theorem)**.** *Let $\mathcal{C}$ be a stable ∞-category with sequential limits. The associated graded functor*

$$\mathrm{gr} \colon \mathrm{Fil}(\mathcal{C}) \longrightarrow \mathrm{gr}(\mathcal{C})$$

factors through the forgetful functor

$$\mathrm{Ch}^\bullet(\mathcal{C}) \longrightarrow \mathrm{gr}(\mathcal{C})$$

and induces an equivalence

$$\mathrm{Fil}^{\mathrm{cpl}}(\mathcal{C}) \simeq \mathrm{Ch}^\bullet(\mathcal{C})$$

between the ∞-category of complete decreasing filtrations in $\mathcal{C}$ and the ∞-category of coherent cochain complexes in $\mathcal{C}$.

Here $\mathrm{Fil}^{\mathrm{cpl}}(\mathcal{C})$ denotes the full subcategory of complete decreasing filtrations. This equivalence provides a canonical definition of convolution.

Definition A.1.3. For a complex $C^\bullet \in \mathrm{Ch}^\bullet(\mathcal{C})$, define its convolution, or realization, by the composite

$$\mathrm{Ch}^\bullet(\mathcal{C}) \simeq \mathrm{Fil}^{\mathrm{cpl}}(\mathcal{C}) \xrightarrow{|\cdot|} \mathcal{C},$$

where $|X| := \mathrm{colim}_{s \in \mathbb{Z}^{\mathrm{op}}} X^s$.

A.1.1 Monoidal structure

From now on, assume that $\mathcal{C}$ is a stable monoidal category that admits sequential colimits and whose tensor product preserves colimits in each variable.

The addition map

$$+ : \mathbb{Z}^{\mathrm{op}} \times \mathbb{Z}^{\mathrm{op}} \longrightarrow \mathbb{Z}^{\mathrm{op}}$$

endows $\mathrm{Fil}(\mathcal{C})$ with the Day convolution monoidal structure. Namely, for $X, Y \in \mathrm{Fil}(\mathcal{C})$, we define

$$X \otimes_{\mathrm{Fil}} Y := +_!(X \boxtimes Y),$$

where $+_!$ denotes left Kan extension along $+$. Therefore

$$(X \otimes_{\mathrm{Fil}} Y)^s \simeq \operatorname*{colim}_{p+q \geq s} X^p \otimes Y^q.$$

The inequality follows from the opposite order.

Lemma A.1.4. *The realization functor*

$$|-| : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$$

is monoidal. In other words, there is a canonical equivalence

$$|X \otimes_{\mathrm{Fil}} Y| \simeq |X| \otimes |Y|.$$

Proof. By definition,

$$|X \otimes_{\mathrm{Fil}} Y| \simeq \operatorname*{colim}_{s \in \mathbb{Z}^{\mathrm{op}}} (X \otimes_{\mathrm{Fil}} Y)^s.$$

Using the formula for Day convolution, this becomes

$$|X \otimes_{\mathrm{Fil}} Y| \simeq \operatorname*{colim}_{s \in \mathbb{Z}^{\mathrm{op}}} \operatorname*{colim}_{p+q \geq s} X^p \otimes Y^q.$$

Let

$$I := \{(p, q, s) \in \mathbb{Z}^3 \mid p + q \geq s\}$$

with the order induced from $(\mathbb{Z}^{\mathrm{op}})^3$, and let

$$P := \mathbb{Z}^{\mathrm{op}} \times \mathbb{Z}^{\mathrm{op}}.$$

The above iterated colimit is the same as

$$\operatorname*{colim}_{(p,q,s) \in I} X^p \otimes Y^q.$$

Let

$$\pi : I \to P, \qquad (p, q, s) \mapsto (p, q).$$

We claim that π is final. Fix $(a, b) \in P$. The category

$$I_{(a,b)/} := I \times_P P_{(a,b)/}$$

has objects triples (p, q, s) satisfying

$$a \geq p, \qquad b \geq q, \qquad p + q \geq s.$$

This category has an initial object, namely

$$(a, b, a + b).$$

Indeed, for every object $(p, q, s) \in I_{(a,b)/}$, we have

$$a \geq p, \qquad b \geq q, \qquad a + b \geq p + q \geq s,$$

so there is a morphism $(a, b, a + b) \to (p, q, s)$ in the induced opposite order. Thus $I_{(a,b)/}$ has contractible nerve. Therefore π is final, and hence

$$\operatorname*{colim}_{(p,q,s)\in I} X^p \otimes Y^q \simeq \operatorname*{colim}_{(p,q)\in P} X^p \otimes Y^q.$$

Since $\otimes$ preserves colimits separately,

$$\operatorname*{colim}_{p,q} X^p \otimes Y^q \simeq \left(\operatorname*{colim}_{p} X^p\right) \otimes \left(\operatorname*{colim}_{q} Y^q\right).$$

Therefore

$$|X \otimes_{\mathrm{Fil}} Y| \simeq |X| \otimes |Y|.$$

□

A.2 $\mathbb{P}^n$-functors and their twists

Let

$$F : \mathcal{C} \to \mathcal{D}$$

be an exact functor between stable ∞-categories. Assume that F has both adjoints

$$L \dashv F \dashv R.$$

The monad RF is an algebra object in the monoidal category $\mathrm{End}(\mathcal{C})$.

Definition A.2.1. A $\mathbb{P}^n$-structure on RF consists of

1. an autoequivalence $H \in \mathrm{Aut}(\mathcal{C})$ such that $H^{n+1}(\ker F) = \ker F$;
2. an algebra object
$$M \in \mathrm{Alg}\big(\mathrm{Fil}(\mathrm{End}(\mathcal{C}))\big).$$
Write $M_i := M^{-i}$ and $\mathrm{gr}_i M := \mathrm{gr}^{-i} M$. We require
$$M_i \simeq 0 \quad (i<0), \qquad M_0 \simeq \mathrm{id}_{\mathcal{C}}, \qquad M_i \simeq RF \quad (i \geq n),$$
and that the map
$$M_0 \longrightarrow M_n \simeq RF$$
corresponds to the unit of the monad
$$\eta : \mathrm{id}_{\mathcal{C}} \longrightarrow RF.$$
3. equivalences
$$\mathrm{gr}_i M \simeq H^i, \qquad 0 \leq i \leq n,$$
compatible with composition $H^i \circ H^j \simeq H^{i+j}$, truncated in degrees $> n$; all other graded pieces vanish;
4. an equivalence $R \simeq H^n L$.

Let
$$\varepsilon : FR \longrightarrow \mathrm{id}_{\mathcal{D}}$$
be the counit of the adjunction $F \dashv R$. Consider the natural transformation
$$\gamma_F : FRFR \longrightarrow FR$$
defined by
$$\gamma_F := FR\varepsilon - \varepsilon FR.$$
Here the difference is taken in the stable ∞-category $\mathrm{End}(\mathcal{D})$.

Using the identification
$$FRFR \simeq F(RF)R,$$
we view γ_F as a morphism
$$FM_nR \longrightarrow FR.$$

By the unit compatibility above, the restriction of γ_F to the degree zero part

$$FM_0R \simeq FR$$

is null-homotopic. Indeed, it is the difference of the two maps

$$FR \xrightarrow{F\eta R} FRFR \rightrightarrows FR,$$

both of which are homotopic to id_{FR} by the triangle identities.

Restrict γ_F along the structure map $FM_1R \to FM_nR$. Since its restriction to FM_0R is null-homotopic, the resulting map factors through the cofiber

$$\mathrm{cofib}(FM_0R \longrightarrow FM_1R) \simeq F\,\mathrm{gr}_1\,MR \simeq FHR.$$

We denote the induced morphism by

$$\psi : FHR \longrightarrow FR.$$

Definition A.2.2. The $\mathbb{P}^n$-twist associated to F is the endofunctor

$$T_F \in \mathrm{End}(\mathcal{D})$$

defined as the convolution

$$T_F := |FHR \xrightarrow{\psi} FR \xrightarrow{\varepsilon} \mathrm{id}_{\mathcal{D}}\,|.$$

The left adjoints of the three terms in the defining complex are

$$(FHR)^L \simeq FH^{-1}L, \qquad (FR)^L \simeq FL, \qquad (\mathrm{id}_{\mathcal{D}})^L \simeq \mathrm{id}_{\mathcal{D}}\,.$$

Taking left adjoints reverses the arrows. Hence the left adjoint of T_F is given by the realization

$$T_F^L \simeq |\,\mathrm{id}_{\mathcal{D}} \longrightarrow FL \longrightarrow FH^{-1}L|,$$

where the first map is the unit of the adjunction $L \dashv F$, and the second map is the left adjoint of

$$\psi : FHR \longrightarrow FR.$$

The Day convolution above induces a monoidal structure on coherent cochain complexes. Let $\mathcal{V}$ be a stable monoidal ∞-category to which the previous discussion applies. Following [Ari21, Definition 6.7, Remark 6.8, Corollary 6.10], we write

$$\otimes_{\mathrm{cc}}$$

for the tensor product on coherent cochain complexes induced from the completed Day convolution product on filtered objects. Thus, for bounded $K^\bullet, L^\bullet \in \mathrm{Ch}^\bullet(\mathcal{V})$, one has

$$(K^\bullet \otimes_{\mathrm{cc}} L^\bullet)^m \simeq \bigoplus_{p+q=m} K^p \otimes L^q.$$

Moreover, realization is monoidal:

$$|K^\bullet \otimes_{\mathrm{cc}} L^\bullet| \simeq |K^\bullet| \otimes |L^\bullet|.$$

When $\mathcal{V} = \mathrm{End}(\mathcal{D})$, the monoidal product is composition of endofunctors, so the formula becomes

$$(K^\bullet \otimes_{\mathrm{cc}} L^\bullet)^m \simeq \bigoplus_{p+q=m} K^p \circ L^q.$$

Theorem A.2.3 (Anno–Logvinenko [AL25]). *The $\mathbb{P}^n$-twist T_F is an autoequivalence of $\mathcal{D}$.*

A direct proof in the present language is omitted.

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