Algebraic K-Theory and Topological Hochschild Homology

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Status. These notes retain the basic infinity-category and Tate-construction arguments and the compactly generated duality calculation. The continuous K-theory, precompact ideals and rigidity programme remains in supporting pending a complete hypotheses and proof audit.

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Chapter 1

Preliminaries

In this note, all categories are $(\infty, 1)$-categories.

1.1 ∞-Categories

Definition 1.1.1. *A quasi-category or an ∞-category $\mathcal{C}$ is a weak Kan complex, i.e. it is a simplical set such that for all $0 < i < n$, every map of simplical sets $\sigma_0 \colon \Lambda^n_i \to \mathcal{C}$ can be extended to a map $\sigma \colon \Delta^n \to \mathcal{C}$.*

Example 1.1.2. *Every Kan complex is an ∞-category, we will denote the ∞-category of Kan complex as $\mathcal{S}$.*

For every ordinary category $\mathcal{C}$, its nerve $N_\bullet(\mathcal{C})$ is an ∞-category. We will abuse terminology by identifying an ordinary category with its nerve.

1.2 (Un-)straightening equivalence

Theorem 1.2.1. *For any ∞-category $\mathcal{C}$, there exists an equivalence given by straightening/unstraightening*

$$\mathrm{coCart}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty)$$

where $\mathrm{coCart}(\mathcal{C})$ is the category of coCartesian fibrations over $\mathcal{C}$.

1.3 Localization of ∞-Categories

Theorem 1.3.1. *Let $\mathcal{C}$ be a small, stable ∞-category and $\mathcal{D} \subset \mathcal{C}$ a stable subcategory.*

1. *Let W be the collection of all arrows in $\mathcal{C}$ whose cofibre lies in $\mathcal{D}$, then the Dwyer-Kan localization $\mathcal{C}/\mathcal{D} := \mathcal{C}[W^{-1}]$ is a stable ∞-category, and $\mathcal{C} \to \mathcal{C}/\mathcal{D}$ is an exact functor.*

   For any stable ∞-category $\mathcal{E}$, the functor $\mathcal{C} \to \mathcal{C}/\mathcal{D}$ induces an equivalence between $\mathrm{Fun}^{ex}(\mathcal{C}/\mathcal{D}, \mathcal{E})$ and the full subcategory of $\mathrm{Fun}^{ex}(\mathcal{C}, \mathcal{E})$ consisting of those functors which send all objects of $\mathcal{D}$ to 0.

2. *Let $X, Y \in \mathcal{C}$ with image $\bar{X}, \bar{Y} \in \mathcal{C}/\mathcal{D}$. The mapping space in $\mathcal{C}/\mathcal{D}$ is given by*

$$\mathrm{Map}_{\mathcal{C}/\mathcal{D}}(\bar{X}, \bar{Y}) \simeq \mathrm{colim}_{Z \in \mathcal{D}_{/Y}} \mathrm{Map}_{\mathcal{C}}(X, \mathrm{cofib}(Z \to Y)),$$

   where the colimit is filtered.

3. *Assume that $\mathcal{E}$ is a presentable stable ∞-category. Then the inclusion $\mathrm{Fun}^{ex}(\mathcal{C}/\mathcal{D}, \mathcal{E}) \subset \mathrm{Fun}^{ex}(\mathcal{C}, \mathcal{E})$ is right adjonit to a localization.*

Proof. (2): Consider the inclusion $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{S}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{S})$, then it admits a left adjonit given by localization functor

$$L\colon \mathrm{Fun}(\mathcal{C}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{S}).$$

Lemma 1.3.2. *Let $X \in \mathcal{C}$ with image $\bar{X} \in \mathcal{C}[W^{-1}]$, localization gives the functor*

$$\mathrm{Map}_{\mathcal{C}[W^{-1}]}(\bar{X}, -) \simeq L(\mathrm{Map}_{\mathcal{C}}(X, -)).$$

Moreover, the following diagram

$$\begin{array}{ccc} \mathcal{C}^{op} & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathcal{S}) \\ \downarrow & & \downarrow L \\ \mathcal{C}[W^{-1}]^{op} & \longrightarrow & \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{S}) \end{array}$$

is commutative, where the horizontal maps are the Yoneda embedding.

Clearly, given any functor $F\colon \mathcal{C} \to \mathcal{S}$, we can form a functor $L(F)\colon \mathcal{C} \to \mathcal{S}$ which sends $X \in \mathcal{C}$ to

$$\mathrm{colim}_{Y \in \mathcal{D}_{/X}} F(\mathrm{cofib}(Y \to X)).$$

It is easy to check that L is the desired functor. □

Chapter 2

Duality for compactly generated stable categories

Definition 2.0.1. *Let $\mathcal{C}$ be a symmetric monoidal category. An object $X \in \mathcal{C}$ is called dualizable if there exists another object $X^\vee$ and morphisms*

$$\mathrm{ev}\colon X^\vee \otimes X \to 1, \quad \mathrm{coev}\colon 1 \to X \otimes X^\vee,$$

such that the compositions

$$\left[X \xrightarrow{\mathrm{coev}\otimes \mathrm{id}} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{id}\otimes \mathrm{ev}} X\right] \simeq \left[X \xrightarrow{\mathrm{id}} X\right],$$
$$\left[X^\vee \xrightarrow{\mathrm{id}\otimes \mathrm{coev}} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{ev}\otimes \mathrm{id}} X^\vee\right] \simeq \left[X^\vee \xrightarrow{\mathrm{id}} X^\vee\right].$$

In particular, if $\mathcal{C}$ is closed and X is dualizable, then $X^\vee$ must be $\underline{\mathrm{Hom}}(X, 1)$.

This standard compact-generation argument is the part of the continuous K-theory background used here; see [Efi24].

Proposition 2.0.2. *For a small stable idempotent-complete category $\mathcal{C}_0$, the category $\mathrm{Ind}(\mathcal{C}_0)$ is dualizable in Pr^L_{st}, with dual $\mathrm{Ind}(\mathcal{C}_0^{op})$.*

Proof. Let $\mathcal{C}_0 \in \mathrm{Cat}^{perf}$, we show that $\mathrm{Ind}(\mathcal{C}_0)$ is dualizable with dual $\mathrm{Ind}(\mathcal{C}_0^{op})$. The Ind-extension of the functor $\mathrm{map}_{\mathcal{C}_0}\colon \mathcal{C}_0^{op} \times \mathcal{C}_0 \to \mathrm{Sp}$, which preserves finite colimits in each variable, gives an evaluation map

$$\mathrm{ev}\colon \mathrm{Ind}(\mathcal{C}_0^{op}) \otimes \mathrm{Ind}(\mathcal{C}_0) \to \mathrm{Sp}\,.$$

Since $\mathrm{Ind}(\mathcal{C}_0) \otimes \mathrm{Ind}(\mathcal{C}_0^{op})$ is stable, a map $\mathrm{Sp} \to \mathrm{Ind}(\mathcal{C}_0) \otimes \mathrm{Ind}(\mathcal{C}_0^{op}) \simeq \mathrm{Ind}(\mathcal{C}_0 \otimes^{Rex} \mathcal{C}_0^{op})$ is equivalent to a single object. Recall that a single object of $\mathrm{Ind}(\mathcal{C}_0 \otimes^{Rex} \mathcal{C}_0^{op})$ is equivalent to a finite limit preserving functor

$(\mathcal{C}_0 \otimes \mathcal{C}_0^{op})^{op} \to \mathrm{An}$, i.e. a functor $\mathcal{C}_0^{op} \times \mathcal{C}_0 \to \mathrm{Sp}$ which preserves finite limit. The coevaluation map $\mathrm{coev}\colon \mathrm{Sp} \to \mathrm{Ind}(\mathcal{C}_0) \otimes \mathrm{Ind}(\mathcal{C}_0^{op})$ can be given by the same map $\mathrm{map}_{\mathcal{C}_0}$ and they obviously satisfies the required conditions.

□

Chapter 3

Tate Construction

3.1 Tate construction for finite groups

Definition 3.1.1. *Let G be a group, and $\mathcal{C}$ an ∞-category. An G-equivariant object in $\mathcal{C}$ is a functor $BG \to \mathcal{C}$, where BG is a fixed classifying space.*

The ∞-category $\mathcal{C}^{BG}$ of G-equivariant objects in $\mathcal{C}$ is the functor ∞-category $\mathrm{Fun}(BG, C)$.

Definition 3.1.2. *Let G be a group and $\mathcal{C}$ an ∞-category.*

1. *Assume that $\mathcal{C}$ admits all colimits indexed by BG. The* homotopy orbits functor *is given by*

$$-_{hG} \colon \mathcal{C}^{BG} \to \mathcal{C} \colon (F \colon BG \to C) \mapsto \mathrm{colim}_{BG} F.$$

2. *Assume that $\mathcal{C}$ admits all limits indexed by BG. The* homotopy fixed points functor *is given by*

$$-^{hG} \colon \mathcal{C}^{BG} \to \mathcal{C} \colon (F \colon BG \to C) \mapsto \lim_{BG} F.$$

Now assume that G is finite, we will construct a norm map $\mathrm{Nm}_G \colon -_{hG} \to -^{hG}$ when the category $\mathcal{C}$ is semiadditive and admits limits and colimits indexed by finite groupoids.[[Lur17, p. 6.1.6.19]]

Construction Let G be a finite group. Then the classifying space BG is a finite groupoid. Let $f\colon BG \to \Delta^0$ be the projection map, then the pullback functor $f^*\colon \mathcal{C}^{\Delta^0} \simeq \mathcal{C} \to \mathcal{C}^{BG}$ admits left and right adjonits $f_!$ and f_*, given by left and right Kan extension along f. ([Lur09, p. 4.3.3.7]) In particualr, we have $f_! = -_{hG}$ and $f_* = -^{hG}$.

Let $\delta\colon BG \to BG \times_{\Delta^0} BG$ be the diagonal map, then we have the norm map $\mathrm{Nm}_\delta\colon \delta_! \to \delta_*$. Since $\mathcal{C}$ is semiadditive, the natural transformation Nm_δ is an equivalence and admits a homotopy inverse Nm_δ^{-1}.

Let $p_0, p_1\colon BG \times_{\Delta^0} BG \to BG$ be the two projections. We have a natural transformation of functors

$$p_0^* \to \delta_*\delta^* p_0^* \simeq \delta_* \xrightarrow{\mathrm{Nm}^{-1}} \delta_! \simeq \delta_!\delta^* p_1^* \to p_1^*$$

which is adjonit to a natural transformation $\beta\colon \mathrm{id}_{\mathcal{C}^{BG}} \to p_{0*}p_1^*$. The pullback diagram of $BG \times_{\Delta^0} BG$ gives an equivalence between $f^*f_* \to p_{0*}p_1^*$, so β determines a map $\mathrm{id}_{\mathcal{C}^{BG}} \to f^*f_*$, which is adjonit to a natural transformation $\mathrm{Nm}_f\colon f_! \to f_*$. We call the map Nm_f as the norm map determined by f.

Let M be a G-equivariant object of $\mathcal{C}$, and we identify M with its image in $\mathcal{C}$. There are canonical maps $M \to M_{hG}$ and $M^{hG} \to M$. Unwinding the definitions, the composition

$$M \to M_{hG} \xrightarrow{\mathrm{Nm}_G} M^{hG} \to M$$

is given by $\sum_{g\in G} \phi_g$, where for each $g \in G$, the map $\phi_g\colon M \to M$ is given by evaliation on the 1-simplex of BG.

Definition 3.1.3. *Let $\mathcal{C}$ be a stable ∞-category which admits all limits and colimits indexed by BG for some finite group G. The Tate construction is the functor*

$$-^{tG}\colon \mathcal{C}^{BG} \to \mathcal{C}\colon X \mapsto \mathrm{cofib}(\mathrm{Nm}_G\colon X_{hG} \to X^{hG}).$$

We will use this definition in particular in the case where $\mathcal{C} = \mathrm{Sp}$. If one takes the Tate spectrum of an Eilenberg-Maclane spectrum HM for a G-module M then one gets back classical Tate cohomology, more precisely $\pi_i(HM^{tG}) \simeq \hat{H}^{-i}(G, M)$.

Assume that G is a normal subgroup of some group H, then for any H-equivariant object $X \in \mathcal{C}^{BH}$, the G-homotopy orbits and G-homotopy fixed points acquire a remaining H/G-action. Consider the projection $f\colon BH \to$

$B(H/G)$, then by [Lur17, p. 6.1.6.3] we have the following commutative diagrams

$$\begin{array}{ccc} \mathcal{C}^{BH} & \xrightarrow{f_!} & \mathcal{C}^{B(H/G)} \\ \downarrow & & \downarrow \\ \mathcal{C}^{BG} & \xrightarrow{-_{hG}} & \mathcal{C} \end{array} \qquad \begin{array}{ccc} \mathcal{C}^{BH} & \xrightarrow{f_*} & \mathcal{C}^{B(H/G)} \\ \downarrow & & \downarrow \\ \mathcal{C}^{BG} & \xrightarrow{-^{hG}} & \mathcal{C} \end{array}$$

where the vertical functors are the forgetful functors. By abuse of notation, we somtimes denote the functor $f_!, f_*$ by

$$-_{hG}, -^{hG} \colon \mathcal{C}^{BH} \to \mathcal{C}^{B(H/G)}.$$

In this case, we still have the Tate construction functor $-^{tG} \colon \mathcal{C}^{BH} \to \mathcal{C}^{B(H/G)}$.

3.2 Tate orbit lemma

Lemma 3.2.1. *Let X be a spectrum with a C_{p^2}-action. Assume that X is bounded below, then we have*

$$(X_{hC_p})^{t(C_{p^2}/C_p)} \simeq 0.$$

Lemma 3.2.2. *Let X be a spectrum with a C_{p^2}-action. Assume that X is bounded above, then we have*

$$(X^{hC_p})^{t(C_{p^2}/C_p)} \simeq 0.$$

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